Moduli space of a certain class of merom connections and its compactification

PLAN

1) (Meromorphic) connections

2) Painlevé V connections and the non-compact moduli space

3) Compactification

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1) Meromorphic connections

Def: A meromorphic connection (E, ∇) is the data of

- $E \to X$ a holomorphic vector bund.
- $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1_X(D)$ a sheaf morphism satisfying the Leibnitz rule

sheaf of holomorphic sections of E

Let us try to better understand the def:

- $E \to X$ hol vector bundle $\Rightarrow$
  $\exists\, U_i$ trivialising open cover s.t. $E_{|U_i} \cong U_i \times \mathbb{C}^r$
  and the transition functions $\{g_{ij}\}$ are holom.

- $\mathcal{E}$ is the sheaf of sections.
  $$\mathcal{E}(U) = \{ \varphi : U \to U \times \mathbb{C}^2 \text{ section of } E \}$$

- $\Omega^1_X(D) = \Omega^1_X \otimes \mathcal{O}_X(D)$ is the "sheaf of differential forms on X with coefficients meromorphic functions on X with poles prescribed by D"

- Leibnitz rule. $\mathcal{E}(U)$ is an $\mathcal{O}_X(U)$-module and it holds that
  $$\nabla(f\varphi) = df \cdot \varphi + f \cdot \nabla\varphi$$

<u>For all the talk</u>: $X = \mathbb{P}^1_{\mathbb{C}}$, $\operatorname{rk} E = 2$

<u>Example</u> (Connection on the trivial bundle)

Let $E := \mathbb{C}^2 \times \mathbb{P}^1$ and (∇, E) a connection.

<u>Fact</u>: There exist a unique ⚠ true only because E is the trivial bundle
$\Omega \in \mathfrak{gl}_2(\Omega^1(D))$ s.t.
$\nabla = d + \Omega$

Since E is trivial, we have global sections and

$$\nabla \Psi = d\Psi + \Omega \Psi = \begin{pmatrix} dy^1 \\ dy^2 \end{pmatrix} + \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \Psi$$

with $\omega_{ij} \in \Omega^1(D) \simeq \Omega^1 \otimes \mathcal{O}(D)$

<u>Fact</u>: Connections on the trivial bundle are system of ODEs $(d\Psi + \Omega\Psi = 0)$

<u>Example</u> (Euler system : $D = [0] + [\infty]$)

$E \cong \mathbb{C}^2 \times \mathbb{P}^1$ trivial bundle and

$$\nabla = d + \Omega = d + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{dx}{x}$$

i.e. $\omega_{11} := \lambda_1 \frac{dx}{x}$, $\omega_{22} = \lambda_2 \frac{dx}{x}$, $\omega_{21} = \omega_{22} = 0$

Since $D = [0] + [\infty]$ we expect ∇ to have a simple pole in 0 and one at $\infty \in \mathbb{P}^1$

We see the pole in 0, to see the one at ∞ we have to change variable on $\mathbb{P}^1$ $x \to \frac{1}{z}$

$$\nabla \rightsquigarrow d + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{d\left(\frac{1}{z}\right)}{1/z} = d - \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{dz}{z}$$

<u>What if $E \to \mathbb{P}^1$ is not trivial??</u>

E is still locally trivial on $\{U_0, U_\infty\}$ cover of $\mathbb{P}^1$

$$\begin{tikzcd} E_{|U_0} \cong \mathbb{C}^2 \times \mathbb{P}^1 \cong E_{|U_\infty} \arrow[from=1-1, to=1-1, phantom] \end{tikzcd}$$

$$E_{|U_0} \underset{g_{0\infty}^{\pm 1}}{\longleftrightarrow} E_{|U_\infty}$$

In particular $\nabla_{|U_0} = d + \Omega_0$, $\nabla_{|U_\infty} = d + \Omega_\infty$

with gluing rule

$$\Omega_0 = g_{0\infty}^{-1} \Omega_\infty g_{0\infty} + g_{0\infty}^{-1} d g_{0\infty}$$

<u>Exemple</u> If $E \cong \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)$ then

$$g_{0\infty} = \begin{pmatrix} x^{k_1} & 0 \\ 0 & x^{k_2} \end{pmatrix}$$

<u>Parallel sections</u> :

A local section $\Psi \in \mathcal{E}(\mathcal{U})$ is ∇-parallel if

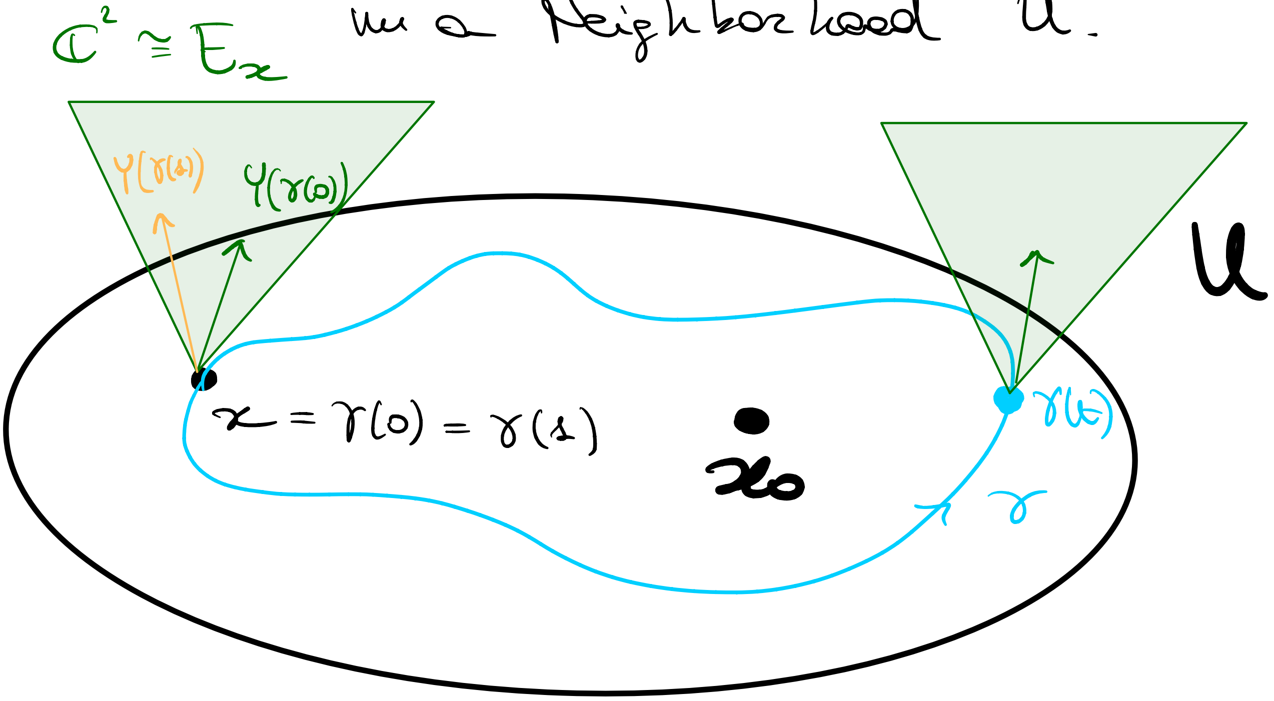
$$\nabla \Psi = 0 = d\Psi + \Omega_{\mathcal{U}} \Psi$$

<u>Thm</u>: For each $\mathcal{U}$ trivialising (∇, E), there exist a basis of ∇-parallel sections

<u>Example</u>: for an Euler system a basis of parallel sections in $\mathcal{U}_0$ is

$$\Psi(x) = x^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} := e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \log(x)}$$

<u>Monodromy</u>: we consider a singularity x_0 and we look what happens in a neighborhood $\mathcal{U}$.



<u>Example</u>: for the singularity $x_0 = 0$ in the Euler system we have the

solution matrix $Y(x) = e^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\log(x)}$

Let us consider x close to zero and the path $\gamma(t) = x e^{2i\pi t}$, $\gamma(0) = \gamma(1) = t$

But $Y(\gamma(0)) = e^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\log(x)}$

$$Y(\gamma(1)) = e^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\log(x e^{2i\pi})} =$$

$$= e^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}(\log(x) + 2i\pi)} =$$

$$= e^{\begin{pmatrix}2i\pi\lambda_1 & 0\\ 0 & 2i\pi\lambda_2\end{pmatrix}} Y(\gamma(0))$$

2 Painlevé V connections and the non-compact moduli space

In order to define a moduli space, we need a suitable equivalence relation.

Def: Two connections $(\nabla, E) \sim (\nabla', E)$ are said <u>equivalent</u> if there exists $H \in Aut(E)$ (holomorphic bundle automorphism)

called gauge equivalence

sending ∇-horizontal section into ∇'-horizontal sections.

i.e. if $\nabla Y = 0$, then $\nabla'(HY) = 0$

Fact: a straightforward calculation in coordinates shows that in each open trivialising set it holds that

$$\Omega = H^{-1}\Omega' H + H^{-1} dH$$

Fact: Equivalent connections have conjugated monodromy

Goal: We want to study equivalent connections for a fixed degree of E $\left(E \cong \mathcal{O}(k_1) \oplus \mathcal{O}(k_2),\ \deg E = k_1 + k_2\right)$, fixed number and type of poles and fixed residual spectrum of Ω

technical reasons arising from the Riemann Hilbert correspondence

<u>Fact</u> under the assumptions above,
all Euler systems are equivalent.
↳ they are "rigid"

<u>Fact</u>: If $\deg D \leq 3$, under the assumptions above, all connections are equivalent.

$\rightsquigarrow$ In order to have a positive dimensional moduli space we have to consider $\deg D \geq 4$

$$\begin{tikzcd}
 & & 2+2 \arrow[dr] & \\
1+1+1+1 \arrow[r] & \overset{P_{V}}{1+1+2} \arrow[ur] \arrow[dr] & & 4 \\
 & & 1+3 \arrow[ur] &
\end{tikzcd}$$

FIX DEGREE 2

Def. A P_{V} connection is $(\nabla, \mathcal{O}\oplus\mathcal{O}(2))$
with polar divisor $D = [0] + 2[1] + [\infty]$

we do not put here the apparent sing $[q]$ (see below)

Up to equivalence we have that any P_{VI} connection can be put in such a normal form:

$$d + \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} 0 & -1 \\ 0 & k_1 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & 1 \\ 0 & k_0 \end{pmatrix} \frac{dx}{x}$$

$$+ \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix} x\,dx + \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix} dx$$

- k_1, k_0, ρ are residual spectral data (fixed)
- $H = H(t,q,p)$ is a quantity that allows $x = q$ to be apparent singularity
- (t,q,p) are the free parameters and hence the coordinate of the moduli space.

RMQ: $x = q$ is a singularity with trivial monodromy $\Rightarrow$ apparent sing

Interpretation of the parameters

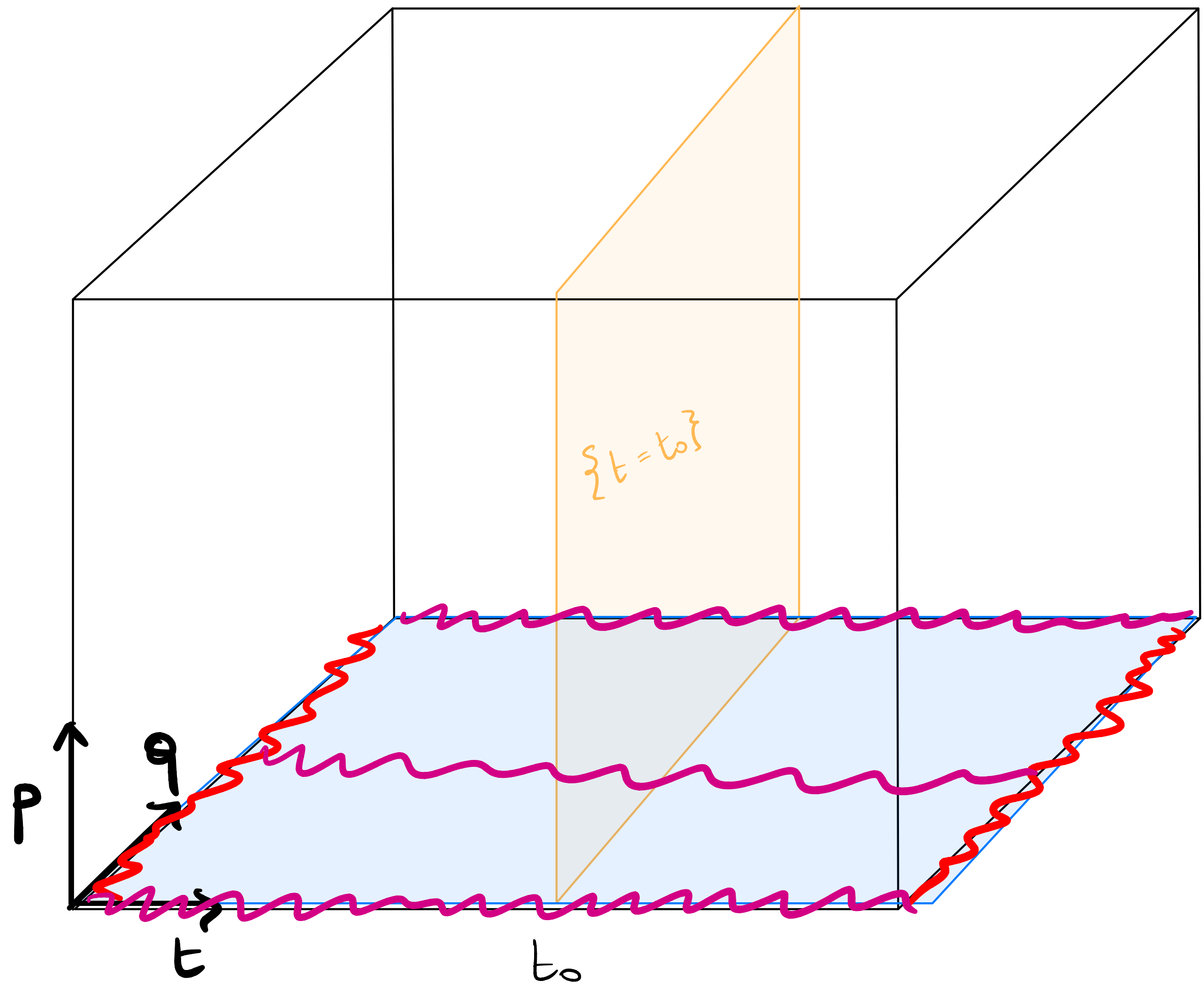
- $(1, t)$ is the eigenvector of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix}$ relative to e.v. t

$t \in \mathbb{C} \setminus \{0\}$

- $(1, p)$ is the eigenvector of the matrix $\begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix}$ relative to e.v. -1

$p \in \mathbb{C}$

- $q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the position of the apparent singularity



Fact: the "non-compact moduli space" is a trivial line bundle $Q \times \mathbb{C}$

where $Q := \mathbb{P}^1_t \times \mathbb{P}^1_q \setminus \{ \{q = 0, 1, \infty\}, \{t = 0, \infty\} \}$

Rmk: • Q comes from the locus where the coordinates t and q lives.

• the line bundle is trivial since the coordinate p is everywhere well defined. Equivalently, $Q \times \mathbb{C}$ has $p = 1$ as a global section.

Let us study a vertical section $\{t = t_0\}$.

In order to do that let us observe that a connection (∇, E) induce a differential equation on the projectivised bundle $\mathbb{P}E$.

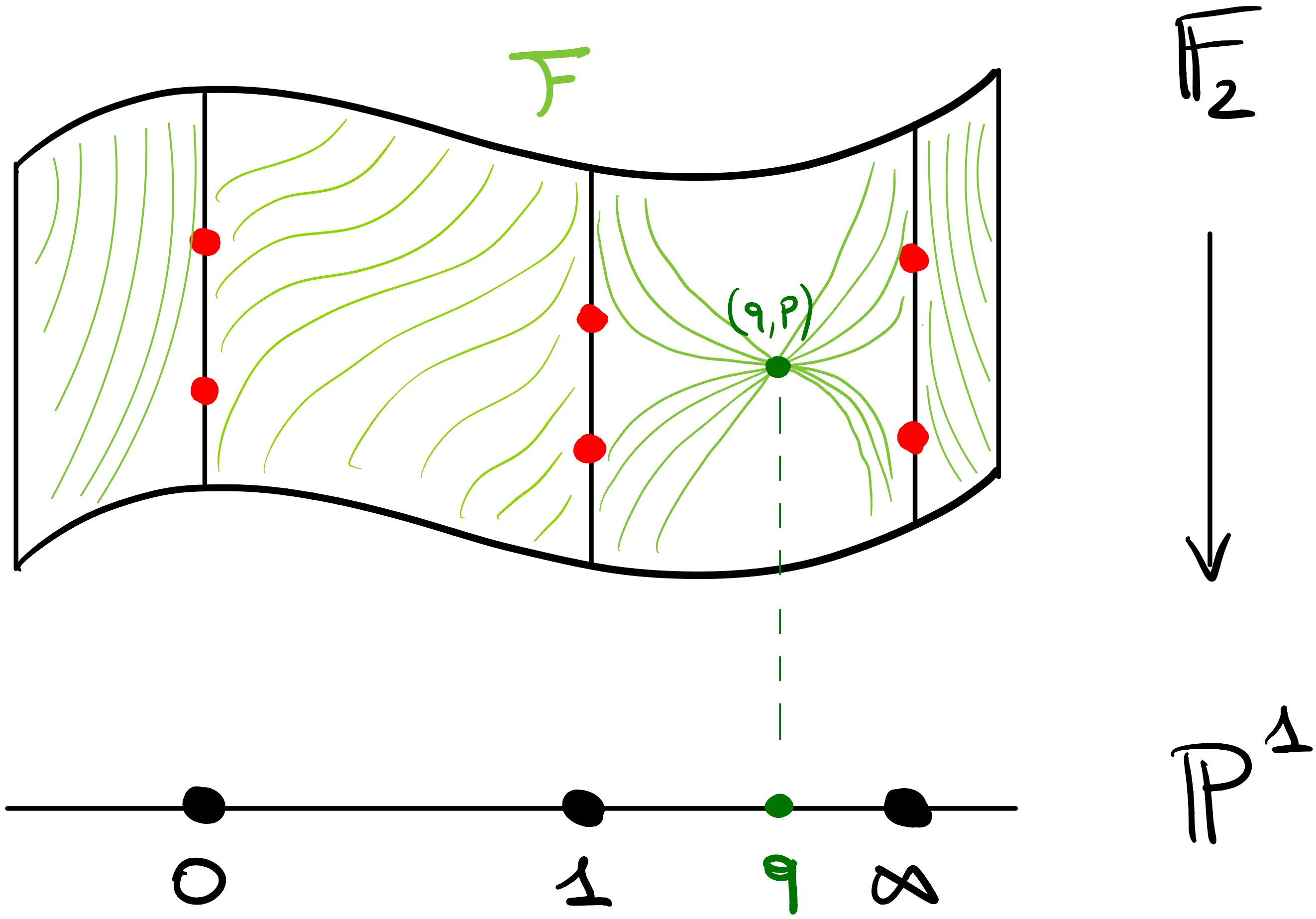
Recall that $\operatorname{rk} E = 2$ and hence $\mathbb{P}E$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$

In our case $E \simeq \mathcal{O} \oplus \mathcal{O}(2)$ and hence

$\mathbb{P}E \cong \mathbb{F}_2$ the second Hirzebruch surface.

The linear system of ODEs induced by ∇ is well defined in the projectivisation and $\nabla\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \iff y' + ay^2 + by + c = 0$ with $y := y_2/y_1$ the projective coordinate.

The solution of the differential equation induce a foliation on $\mathbb{F}_2$:



Where $(q,p) \in \mathbb{F}_2$ is the only radial singularity of the foliation.

RMK : The monodromy of ∇ is the same as the monodromy of the foliation and radial singularities have no monodromy.

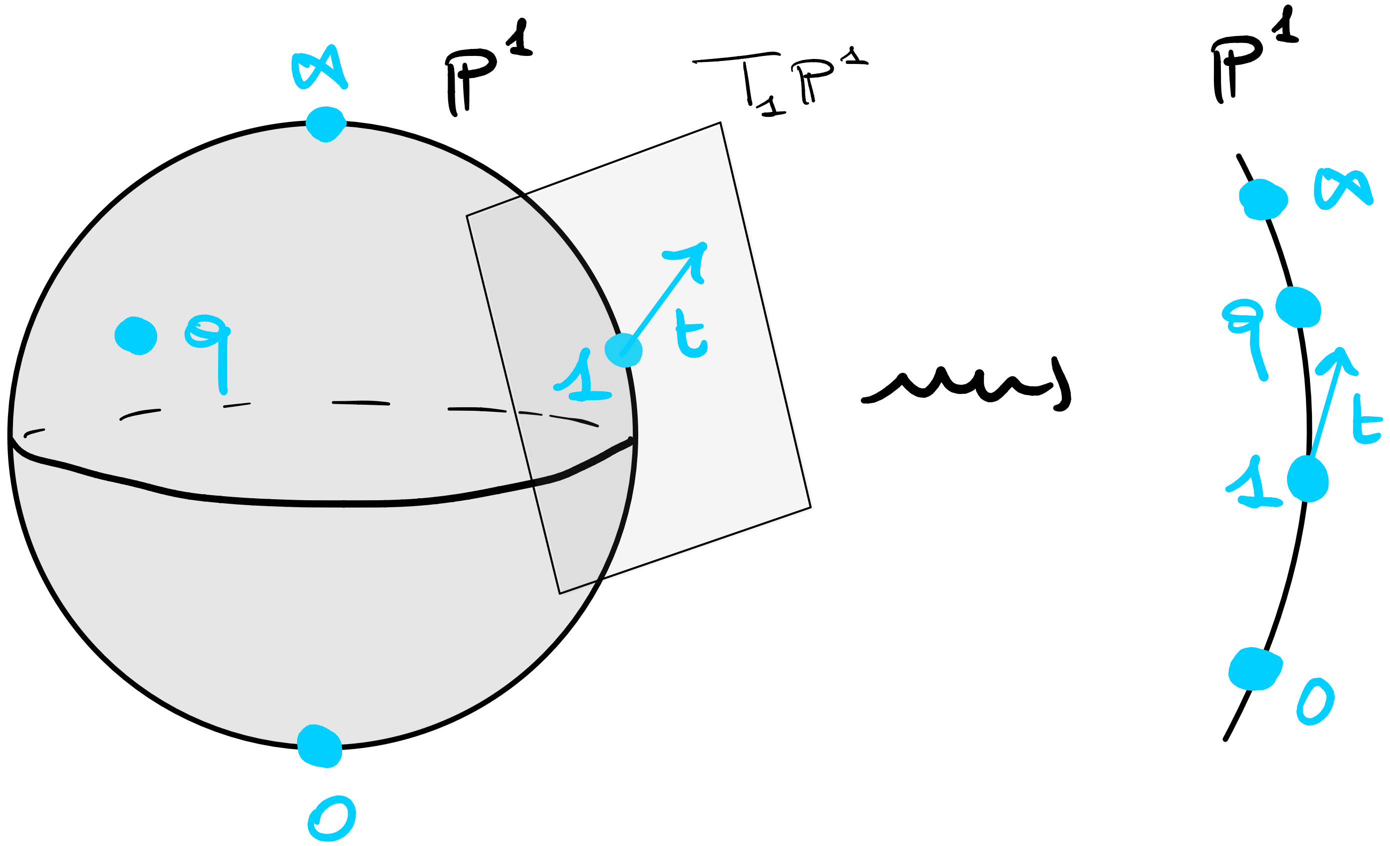
3) Compactification

We would like to allow $t = 0, \infty$ and $q = 0, 1, \infty$.

To do so, we can use a similar approach to the Deligne - Mumford compactification of the spaces $M_{0,n}$ via stable curve.

Looking at the action of any $f \in \mathrm{Aut}(\mathbb{P}^1)$ on the connection, we see that we can think of $t \in T_1\mathbb{P}^1$.

Hence we want to study stable curves for this model

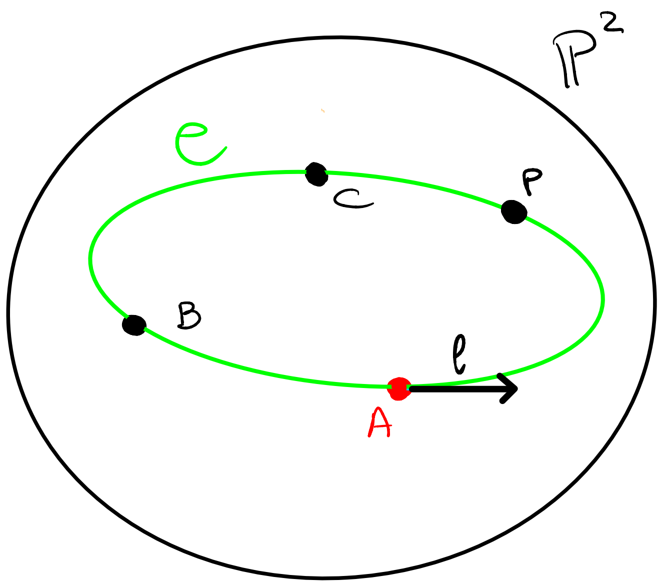


Following the Deligne-Mumford idea, we can consider the moduli space of rational marked curves in $\mathbb{P}^2$ passing through three given points A, B, C in general position and having in, for instance, A a prescribed tangent direction ℓ s.t. ℓ is not parallel to the lines $\overline{AB}$ or $\overline{AC}$.

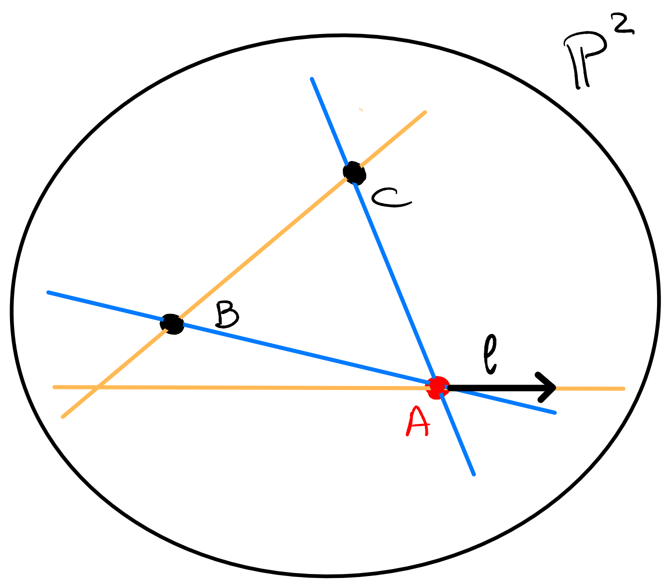
Since it is well known that there exist only one smooth conic passing through four point in general position in $\mathbb{P}^2$ having a prescribed tangent direction at one of them, the moduli space of such conics is

$$\mathbb{P}^2 \setminus \{\overline{AB}, \overline{AC}, \overline{BC}, \ell\}.$$

Indeed, once fixed A, B, C, ℓ the lines $\{\overline{AB}, \overline{AC}, \overline{BC}, \ell\}$ correspond exactly to the points that are not in general position w.r.t. them.



Smooth conic corresponding to the point P

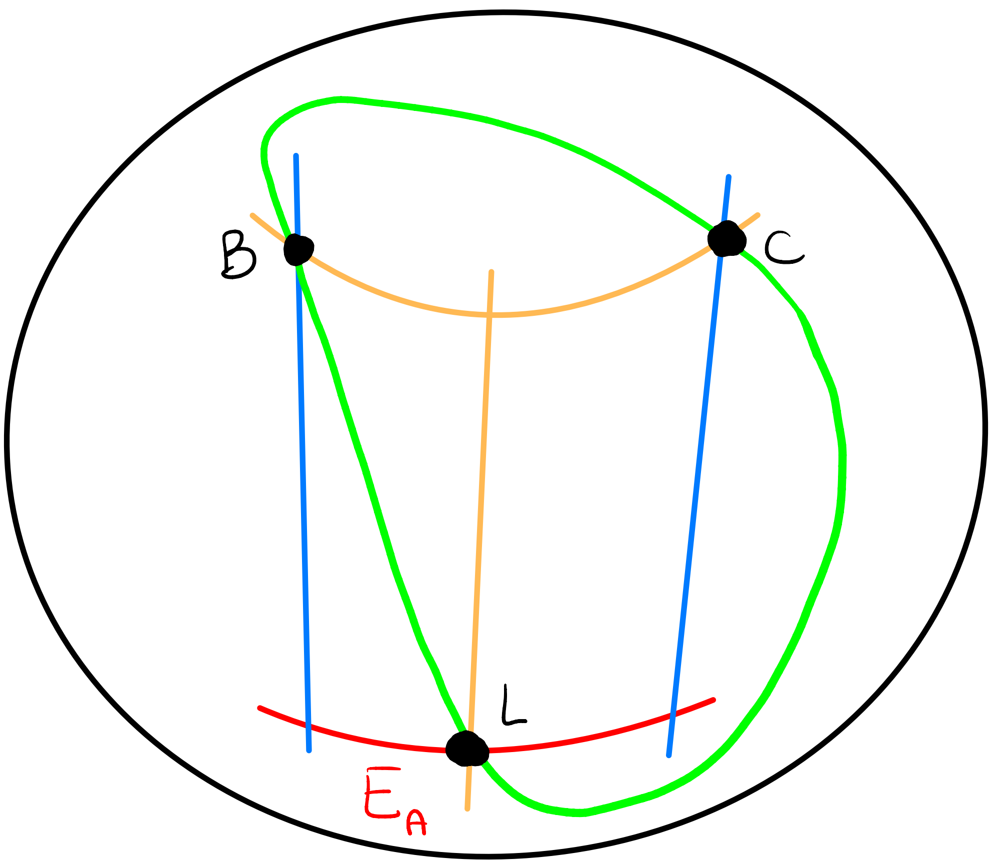


The two singular conics

We want to compactify this moduli space. Roughly speaking, we want to see what happens when we choose points on $\{\overline{AB}, \overline{AC}, \overline{BC}, \ell\}$

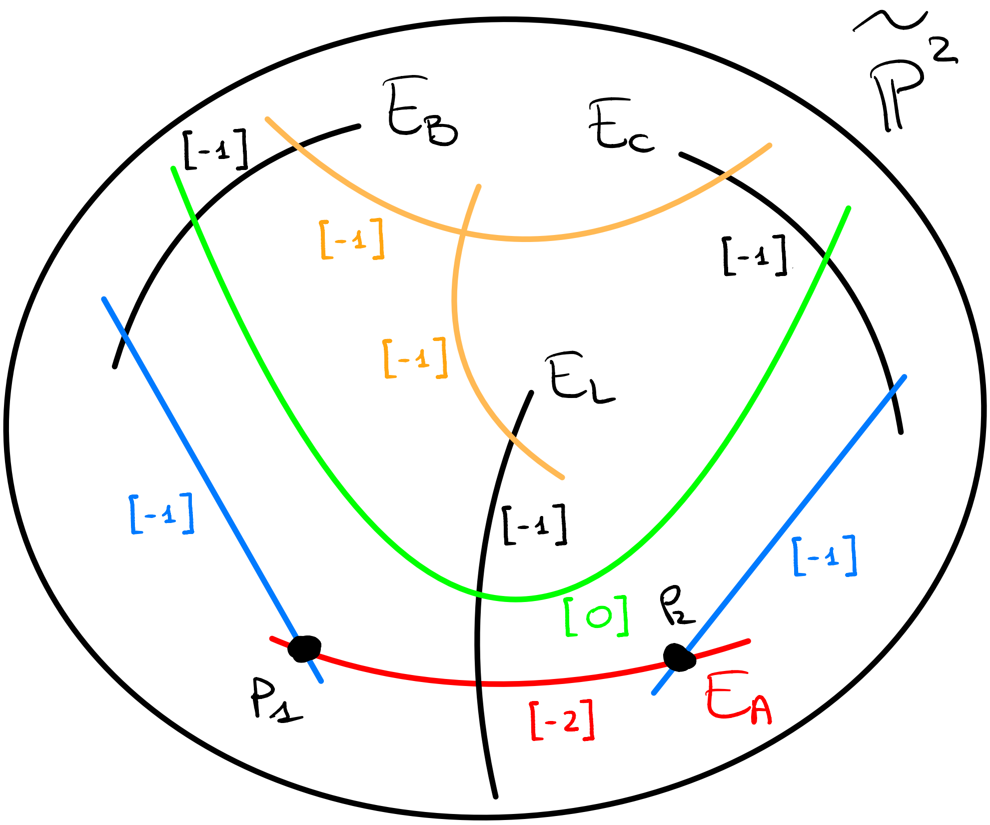
First step: we can add to $\mathcal{C}$ the points lying on $\overline{AB}\setminus\{A,B\}$, $\overline{AC}\setminus\{A,C\}$, $\overline{CD}\setminus\{C,D\}$, $\overline{AD}\setminus\{A\}$ by considering singular conics.

Second step: we can blow-up the point A. Now we will denote by L the point on the exceptional divisor E_A representing ℓ



Third step : we blow-up the points

A, B, L. We denote $\tilde{\mathbb{P}}^2$ the resulting alg. var. and we note for any curve its self-inters number since we will need it later on.



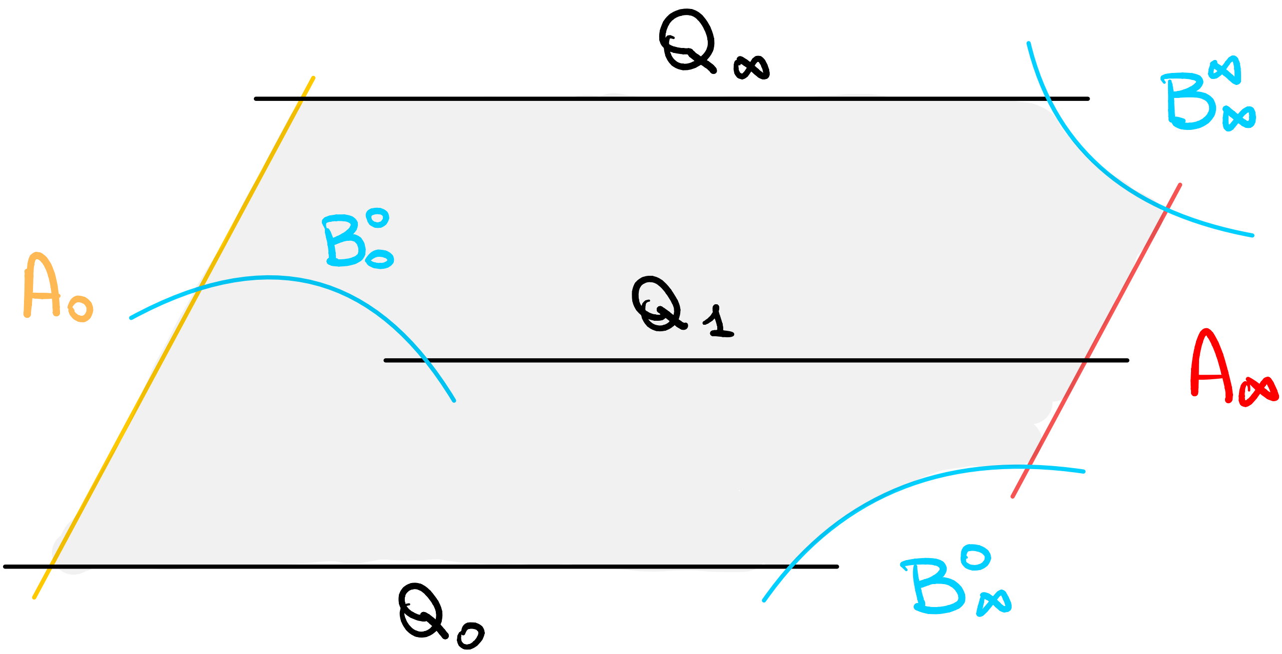
Facts • Each point in $(\tilde{\mathbb{P}}^2 \setminus E_A) \cup \{P_1, P_2\}$ defines a conic in $\mathbb{P}^2$ with the properties we settled

• The conics defined by P_1 and P_2 are isomorphic "by exchanging the two branches"

<u>Consequence</u> : The compact moduli space we want is given by the contraction of E_A in $\widetilde{\mathbb{P}}^2$

We want to come back to our original problem :

<u>Fact</u> $\widetilde{\mathbb{P}}^2 \cong Bl_{(0,\infty),(\infty,0),(\infty,\infty)} \mathbb{P}^1 \times \mathbb{P}^1 := \widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$



<u>Consequence</u> The compact moduli space $\overline{Q}$ we want is given by the contraction of A_∞ in $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$

<u>Recall</u> : The non compact moduli space

is a trivial line bundle over Q.

Goal: Understand how it extends to the compactification $\overline{Q}$

PROP: The line bundle is

$$\mathcal{O}_{\overline{Q}}(D)$$

for $D = A_0 + 2Q_0 + B_\infty^0 - B_\infty^\infty$ ∎