

Compactification of the Painlevé V Foliation

Seminar IMPA

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Meromorphic Connections

MEROMORPHIC CONNECTIONS

Definition

A meromorphic connection (∇, E, D) over a complex manifold X is the data of:

- a holomorphic vector bundle $E \rightarrow X$,
- a morphism $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1(D)$

where D is a effective divisor of X called the polar divisor of ∇ .

Leibnitz Rule

For the \mathcal{O}_X -module structure of \mathcal{E} .

$$\nabla(f\sigma) = df \cdot \sigma + f\nabla\sigma$$

AN EXAMPLE

Euler System: $X = \mathbb{P}^1$ and $\deg D = 2$

$(\nabla, E, D) = (d + \Omega, \mathcal{O} \oplus \mathcal{O}, [0] + [\infty])$.

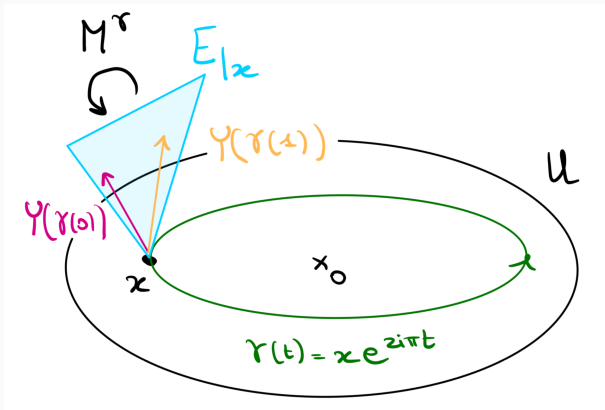
$$d \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \Omega \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} + \underbrace{\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{dx}{x} \right]}_{\Omega} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Has simple poles in $x = 0$ and $x = \infty$.
- Has a solution matrix of the form

$$Y(x) = x^A := \exp(A \log x)$$

- Has non-trivial *monodromy* due to the complex logarithm.
- Any deformation of parameters modify the monodromy (rigidity).

MONODROMY



- Defined by the process of analytic continuation of $Y(x)$
- Only depends on the homotopy class of γ in $U \setminus \{0\}$.

LOCAL FORM

Local form of a connection

Let U be a simply connected open set trivialising E .

$$\nabla_U = d + \Omega_U \quad \text{with} \quad \Omega_U = \begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \in \mathfrak{gl}(\Omega^1(D \cap U))$$

with gluing condition on $U_i \cap U_j$

$$\Omega_i = g_{i,j}^{-1} \cdot \Omega_j \cdot g_{i,j} + g_{i,j}^{-1} \cdot dg_{i,j},$$

where $\{g_{i,j}\}$ is the cocycle of the bundle E .

HORIZONTAL SECTIONS

Horizontal sections

A local section $Y: U \rightarrow U \times \mathbb{C}^2$ is horizontal if :

$$\nabla Y = 0 \iff \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} + \begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

Theorem (Illyashenko, Yakovenko)

Let (∇, E) be a connection on \mathbb{P}^1 and U a simply connected open set of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then:

- There exists a fundamental matrix of solution $Y(x)$ defined everywhere over U .
- Any other matrix of solutions $Y'(x)$ differs from $Y(x)$ by a constant matrix factor $Y'(x) = Y(x) \cdot C$.

EQUIVALENCE OF CONNECTIONS

Gauge equivalence of connections

We say that $(\nabla, E) \sim (\nabla', E)$ if there exists $\Phi \in \text{Aut}(E)$ sending ∇ -horizontal sections in ∇' -horizontal sections.

The connections matrices are related by this equality:

$$\Omega_U = \Phi^{-1}\Omega'_U\Phi + \Phi^{-1}d\Phi$$

Fact

Equivalent connections have conjugated monodromy:

$$(\nabla, E) \sim (\nabla', E) \implies M_\nabla \sim M_{\nabla'}$$

MEROMORPHIC GAUGE TRANSFORMATIONS

Remark

Meromorphic gauge transformations do not change the monodromy and changes the degree of the bundle.

Example

$$\Psi := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x-q} \end{pmatrix}$$

is a meromorphic gauge transformation, then:

$$\Psi : (\nabla, \mathcal{O} \oplus \mathcal{O}, D) \mapsto (\tilde{\nabla}, \mathcal{O} \oplus \mathcal{O}(1), D + [q]),$$

and the pole q has trivial monodromy.

Conclusion

Since we are interested in isomonodromic deformations, we have to study connections up to meromorphic gauge transformations.

Painlevé V Connections

PAINLEVÉ V CONNECTIONS

Painlevé V Connections

Meromorphic connections (∇, E, D) such that

- $X = \mathbb{P}^1$
- $D = [0] + 2[1] + [\infty]$

Remark

Technical convention: the polar divisor D is minimal w.r.t. meromorphic gauge transformations.

Consequence: in D appear only poles with non trivial local monodromy.

NORMAL FORM

Normal Form on $\mathcal{O} \oplus \mathcal{O}(2)$ (Diarra, Loray 2019)

$$\nabla|_0 = d + \Omega_0 =$$

$$d + \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} 0 & -1 \\ 0 & \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & 1 \\ 0 & -\kappa_0 \end{pmatrix} \frac{dx}{x} \\ + \begin{pmatrix} 0 & 0 \\ \kappa_\infty & 0 \end{pmatrix} xdx + \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ \hat{K} & 0 \end{pmatrix} dx,$$

Fixed Parameters

$$\Theta := \{\kappa_0, \kappa_1, \kappa_\infty\} \subseteq \mathbb{C}$$

FREE PARAMETERS

$$\begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2}$$

Description

$$t \in \mathbb{C} \setminus \{0\} \cong T_1\mathbb{P}^1 \setminus \{0\}$$

is the non zero eigenvalue of the matrix of order -2 of the pole $x = 1$.
We can see t as a non zero element of $T_1\mathbb{P}^1$

FREE PARAMETERS

$$\begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q}$$

Description

$$q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

is an apparent singularity that shows during the process of normalisation and

$$\begin{pmatrix} 1 \\ p \end{pmatrix}$$

is the eigenvector relative to the non zero eigenvalue of the matrix coefficient of $\frac{dx}{(x-q)}$.

GEOMETRICAL INTERPRETATION OF THE PARAMETERS

Projectivisation of the Connection

Given a Painlevé V connection $(\nabla, \mathcal{O} \oplus \mathcal{O}(2))$, we can consider

- $\mathbb{P}(E)$, that is a \mathbb{P}^1 -bundle with fiber coordinate $y := y_2/y_1$
- The Riccati differential equation

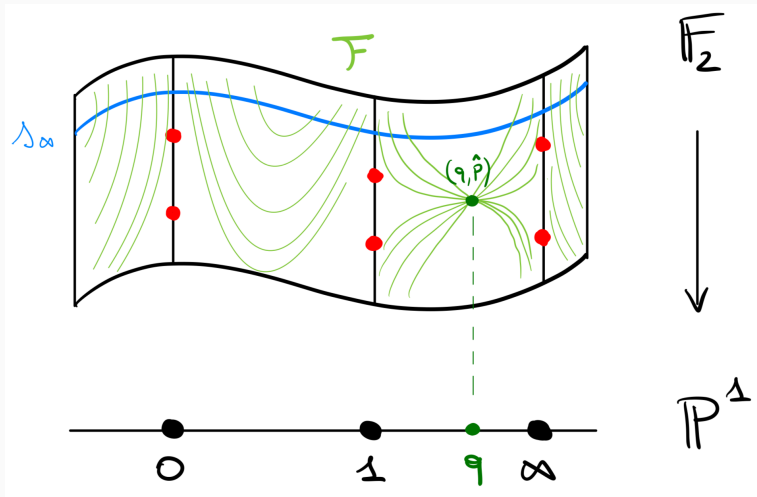
$$dy = -\omega_{1,2}y^2 + \omega_{2,2}y + \omega_{2,1}$$

induced by ∇ .

Riccati Foliation

- $\mathbb{F}_2 := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ the second Hirzebruch surface
- \mathcal{F} the foliation on \mathbb{F}_2 induced by the differential equation below.

GEOMETRICAL INTERPRETATION OF THE PARAMETERS



MONODROMY

Fact

$$\text{Mon}(\nabla, \mathcal{O} \oplus \mathcal{O}(2)) = \text{Mon}(\mathcal{F}, \mathbb{F}_2)$$

Stokes Phenomena

The local monodromy around $x = 1$ is more complicated: solutions are defined only on sectors around the singularity and the change of sector give rise to a monodromy.

MONODROMY

Problem

Isomonodromic deformations

Tools

- Moduli space of connections
- Deligne-Mumford compactification via stable curves
- Painlevé V differential equation

Painlevé Equations (Jimbo, Miwa, Ueno 1981)

Are second order differential equations whose solutions parametrise isomonodromic deformations of rank 2 meromorphic connections with $\deg D = 4$.

Moduli Space and Compactification

NON-COMPACT MODULI SPACE

What to do?

We have to find a suitable algebraic variety of dimension 3 in which our parameters

$$(t, q, p)$$

live.

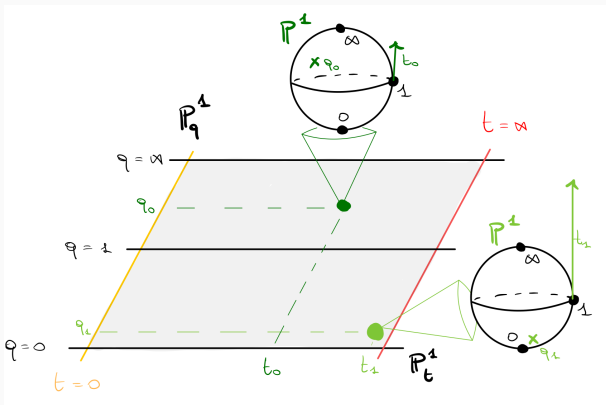
In order to do that we recall that

- $t \in \mathbb{P}^1 \setminus \{0, \infty\} \cong T_1\mathbb{P}^1 \setminus \{0\}$,
- $q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$,
- $p \in \mathbb{C}$.

PARAMETERS (t, q)

Moduli space of stable curves \mathcal{M}

$$\mathcal{M} := \left\{ q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}; t \in T_1\mathbb{P}^1 \setminus \{0\} \right\}.$$



PARAMETER p

Line Bundle

We can add to the picture the parameter $p \in \mathbb{C}$ as the coordinate on the fiber of a trivial line bundle.

Moduli space of Connections $\mathcal{C}on_V^\Theta$

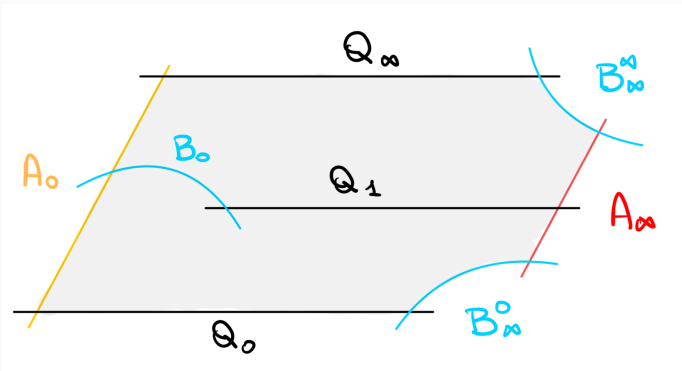
$$\mathcal{C}on_V^\Theta \supseteq \mathcal{M} \times \mathbb{C} \ni (t, q, p)$$

Our moduli space contains as a Zarisky open set the trivial line bundle $\mathcal{M} \times \mathbb{C}$.

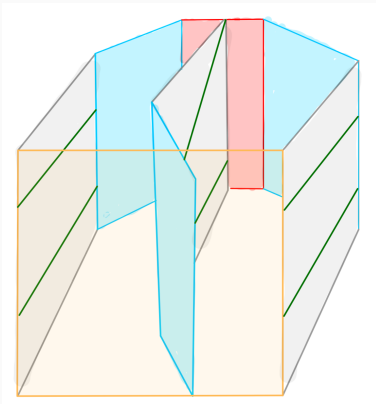
COMPACTIFICATION $\overline{\mathcal{M}}$

Deligne-Mumford

Via a study inspired to the Deligne-Mumford compactification of the moduli space $\mathcal{M}_{g,n}$ via stable curves, we get that our compactification is as in the picture:



COMPACTIFICATION $\overline{\text{Con}}_V^\Theta$



Compactification (M., 2025)

$$\overline{\text{Con}}_V^\Theta \rightarrow \mathcal{O}_{\overline{\mathcal{M}}}(2Q_0 + B_\infty^0 - B_\infty^\infty + A_0) \cup s_\infty$$

Painlevé V Foliation

PAINLEVÉ V EQUATION

The Equation (PV)

$$q''(t) = \left(\frac{1}{2q(t)} + \frac{1}{q(t) - 1} \right) q'(t)^2 - \frac{1}{t} q(t)'$$
$$+ \frac{(q(t) - 1)^2}{t^2} \left(\alpha q(t) + \frac{\beta}{q(t)} \right) + \gamma \frac{q(t)}{t} + \delta \frac{q(t)(q(t) + 1)}{q(t) - 1}$$

Where $\alpha, \beta, \gamma, \delta$ are parameters depending on Θ .

PAINLEVÉ V EQUATION

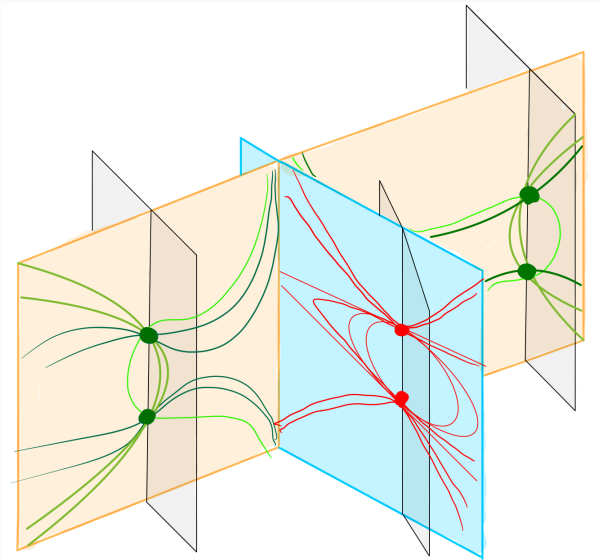
The Hamiltonian System

$$\begin{cases} \frac{\partial H^V}{\partial p} = \frac{dq}{dt} \\ \frac{\partial H^V}{\partial q} = -\frac{dp}{dt} \end{cases}$$

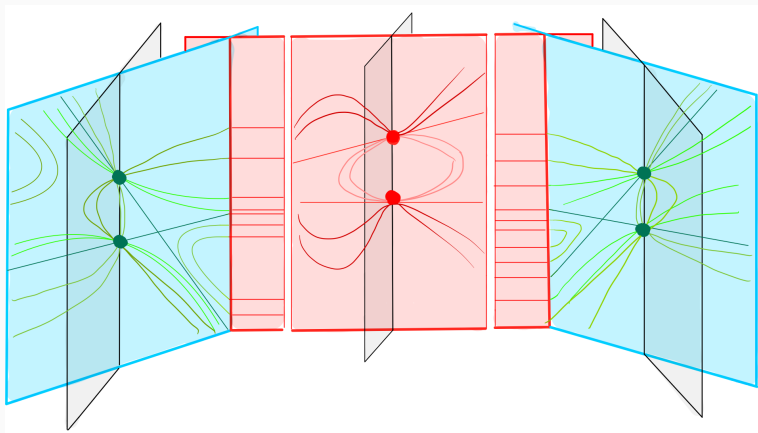
The Hamiltonian Function (Ohyama, 2006)

$$H^V := \frac{q(q-1)^2 p^2 - (\kappa_0(q-1)^2 + \kappa_1 q(q-1) - tq)p + \kappa_\infty(q-1)}{t}$$

FIRST INTEGRALS AROUND $t = 0$



FIRST INTEGRALS AROUND $t = \infty$



GOALS

Short terms goals

- Compute first integrals in a neighbourhood of the hypersurfaces $t = 0, \infty$.
- Understand the singularities of the foliation.
- Eventually apply MMP to have a "good" model.

Long terms goals

- Study the dynamics of the foliation in relation to the dynamics of the wild character variety associated (moduli space of monodromies).
- Apply the same study to other Painlevé equations

THANKS :)

Obrigado pela sua atenção !!