# Compactification of the fifth Painlevé Foliation

Rencontre IsoMoDyn, Rennes

Mattia Morbello

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Université de Rennes

# Meromorphic Connections

# Meromorphic Connections on $\mathbb{P}^1$

## Definition

A rank 2meromorphic connection $(E, D, \nabla)$ on $\mathbb{P}^1$ is the data of:

- a rank 2 holomorphic vector bundle $E \to \mathbb{P}^1$,
- a effective divisor $D$ of $\mathbb{P}^1$ called the polar divisor,
- a morphism $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$.

## Remark

The sheaf $\Omega^1(D) := \Omega^1 \otimes \mathscr{O}(D)$ is the sheaf of meromorphic 1-forms with poles as prescribed by $D$.

## Leibniz Rule

The operator $\nabla$ is required to satisfy

$$\nabla(f\sigma) = df \cdot \sigma + f\nabla\sigma$$

# Local Expression

## Trivialisation

On a trivializing open set $U$, a connection $(E, D, \nabla)$ has the form

$$\nabla_{|U} = d + \Omega_U, \quad \text{for} \quad \Omega_U \in \mathfrak{gl}_2(\Omega^1(D \cap U)).$$

## Gluing conditions

Any connection $(E, D, \nabla)$ on $\mathbb{P}^1$ is then identified with the data

$$\begin{cases} d + \Omega_0 & \text{on } U_0 := \mathbb{P}^1 \setminus \{\infty\} \\ d + \Omega_\infty & \text{on } U_\infty := \mathbb{P}^1 \setminus \{0\} \\ g_{0,\infty} & \text{cocycle of } E \end{cases}$$

with gluing condition on the overlap

$$\Omega_0 = g_{0,\infty}^{-1} \cdot \Omega_\infty \cdot g_{0,\infty} + g_{0,\infty}^{-1} \cdot dg_{0,\infty}.$$

# Equivalence of Connections

## Gauge equivalence of connections

We say that $(E, D, \nabla) \sim (E', D', \nabla')$ if there exists $\Phi \in \mathrm{Mor}(E, E')$ such that

$$\Phi^* \nabla' = \nabla.$$

## Fact

$$(E, D, \nabla) \sim (E', D', \nabla') \quad \Longrightarrow \quad M_\nabla \sim M_{\nabla'}$$

## Local equivalence

The connections matrices on a trivialising open set $U$ are locally related by:

$$\Omega_U = \Phi^{-1} \Omega'_U \Phi + \Phi^{-1} d\Phi$$

# Equivalence of Connections

**Example**

The action of the meromorphic gauge transformation

$$\Phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x-q} \end{pmatrix}$$

on the connection $(\mathcal{O} \oplus \mathcal{O}, D, d + \Omega)$ is

$$(\mathcal{O} \oplus \mathcal{O}(1), D + [q], d + \Omega'),$$

where

$$\Omega' := \begin{pmatrix} \omega_{1,1} & (x-q)\omega_{1,2} \\ \frac{1}{x-q}\omega_{2,1} & \omega_{2,2} - \frac{dx}{x-q} \end{pmatrix}$$

# Painlevé V Connections

# Connexions du type PV

**Definition**

Meromorphic connections $(E, D, \nabla)$ such that

$$D^{\text{min}} = [0] + 2[1] + [\infty],$$

where $D^{\text{min}}$ is the *minimal* polar divisor w.r.t. the gauge equivalence class of $(E, D, \nabla)$.

**Consequence**

In $D^{\text{min}}$ only appear poles

- that cannot be eliminated nor reduced via a gauge transformation,
- with minimal Poincaré rank,
- with non-trivial monodromy.

# Normal Form

## Normal Form on $\mathscr{O} \oplus \mathscr{O}(2)$ (Diarra, Loray 2019)

$$\nabla_{|0} = d + \Omega_0 =$$

$$d + \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} 0 & -1 \\ 0 & \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & 1 \\ 0 & -\kappa_0 \end{pmatrix} \frac{dx}{x}$$

$$+ \begin{pmatrix} 0 & 0 \\ \kappa_\infty & 0 \end{pmatrix} x dx + \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ \hat{K} & 0 \end{pmatrix} dx,$$

## Fixed Parameters

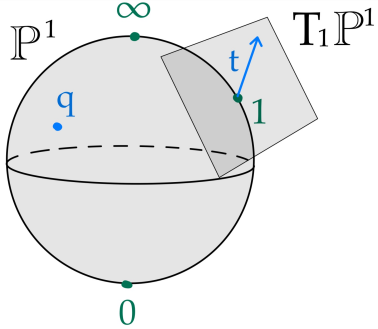
$$\Theta := \{\kappa_0, \kappa_1, \kappa_\infty\} \subseteq \mathbb{C}$$

# Geometrical Interpretation of $t$ and $q$

**Action of $f \in \mathrm{Aut}(\mathbb{P}^1)$**

Let $f \in \mathrm{Aut}(\mathbb{P}^1)$, then:

- $q \mapsto f(q)$,
- $t \mapsto Df(1) \cdot t$.



# Rational Irregular Curves

## Definition

A rational irregular curve is $(\mathbb{P}^1, D, J)$, where

- $D = \sum_{i \in I} n_i [p_i]$ is an effective divisor,
- $J = \{j^{n_i - 1}(p_i)\}_{i \in I}$ is a collection of jets.

## Example

$(\mathbb{P}^1, D, J)$ with

- $D = [0] + 2[1] + [\infty] + [q]$
- $J = \{\, j^0(0),\ j^1(1) = (1, t),\ j^0(\infty),\ j^0(q) \,\}$.

## Proposition

$[(\nabla, E, D)]$ "generic" of PV type $\iff$ a rational irregular curve and $p \in \mathbb{C}$.

# Moduli Space and Compactification

# MODULI SPACE

**Proposition**

$$[(\nabla, E, D)] \text{ "generic" of PV type } \iff \begin{cases} (\mathbb{P}^1, D, J) & \text{, and} \\ p \in \mathbb{C} \end{cases}$$

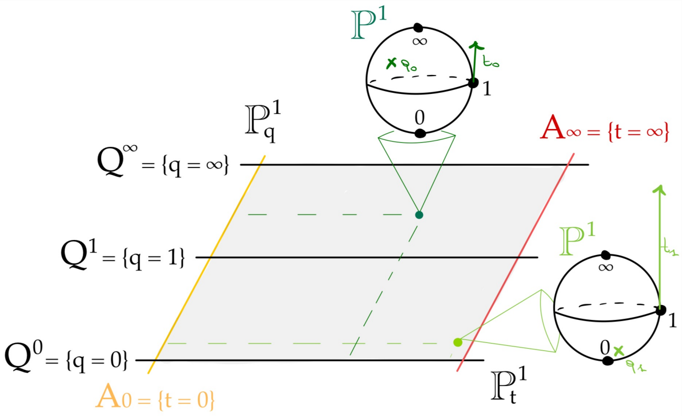
**Moduli space of rational irregular curves $\mathcal{M}$**

$$\mathcal{M} := \left\{ q \in \mathbb{P}^1 \setminus \{0, 1, \infty\};\ \ t \in T_1\mathbb{P}^1 \setminus \{0\} \right\}.$$

**Moduli space of PV connections**

$$\mathfrak{Con}_{\Theta}^{V} \supseteq \mathcal{M} \times \mathbb{C}$$

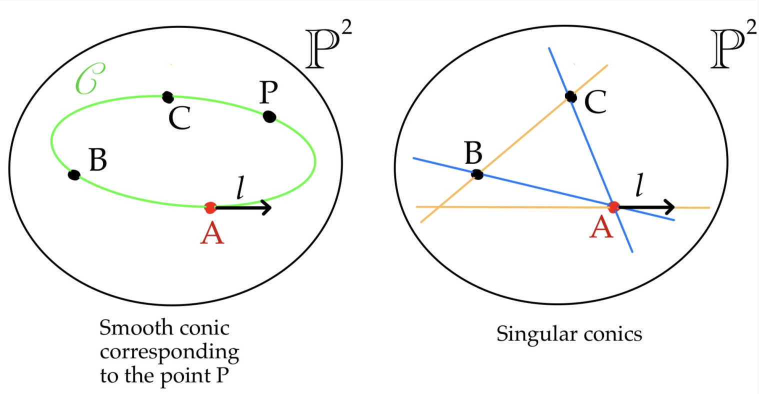
# MODULI SPACE $\mathcal{M} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$




**Remark**

The moduli space $\mathcal{M}$ is a generalization of $\mathcal{M}_{0,5}$ where the fifth marked point is replaced by a tangent vector.

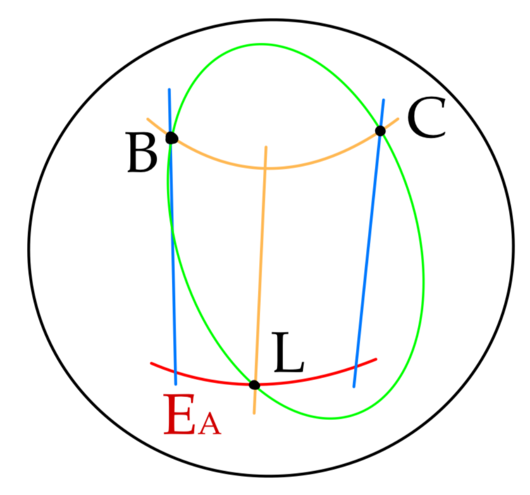
# Moduli Space of Conics



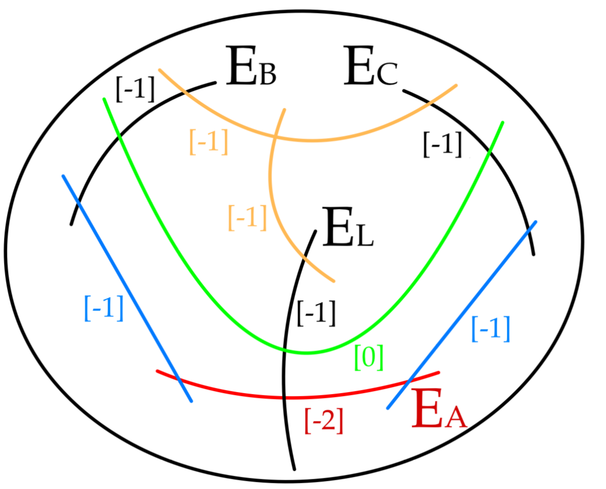

## Moduli Space of Smooth Conics

$$\mathbb{P}^2 \setminus \left(\overline{AB}, \overline{AC}, \overline{BC}, l\right) \cong \mathcal{M}$$

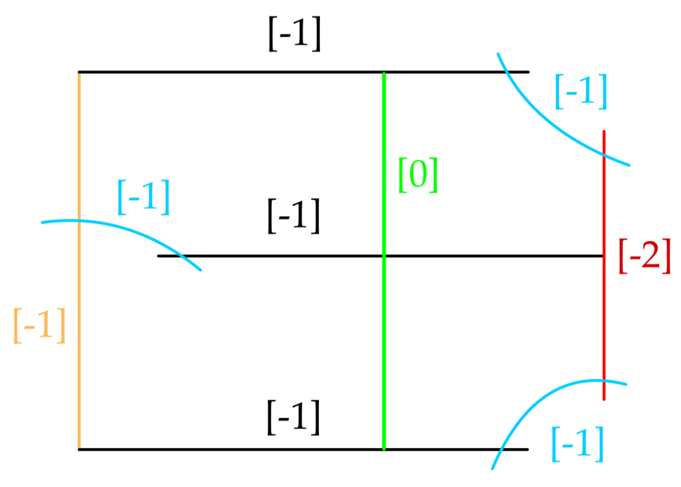
# Blow-up of $A$



# Blow-up of $B$, $C$ and $L$



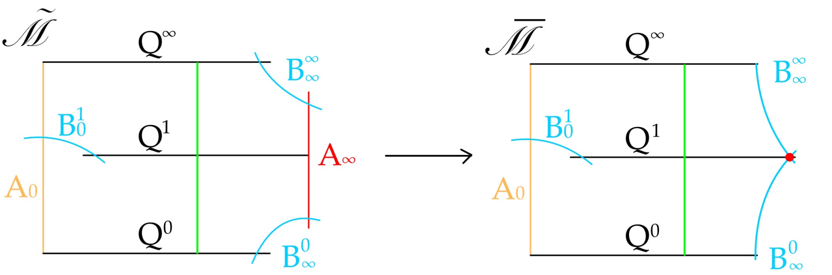
# Compactification $\overline{\mathcal{M}}$



# Compactification $\overline{\mathcal{M}}$

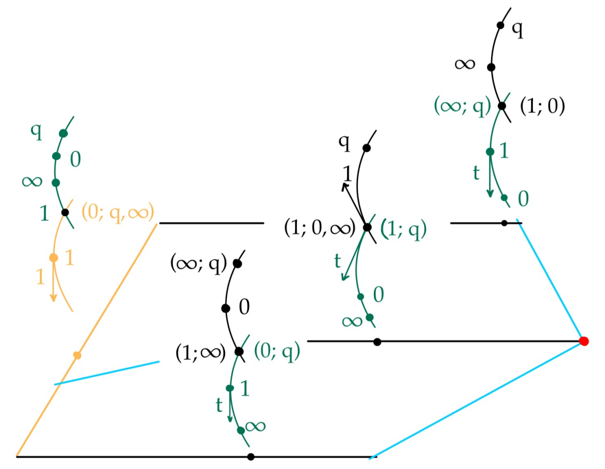
**Compactification (M., 2025)**

Following, *mutatis mutandis*, a similar idea that Kapranov used to compactify the spaces $\mathcal{M}_{0,n}$, we get:




where $\widetilde{\mathcal{M}}$ is the weak del Pezzo surface of degree five.

# Universal Curve



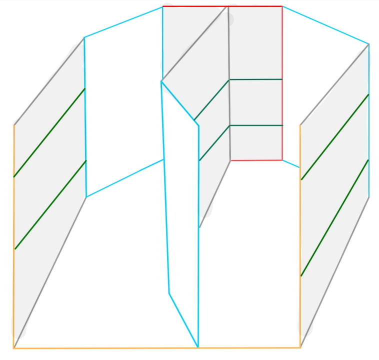
# COMPACTIFICATION $\overline{\mathfrak{Con}_V^{\Theta}}$

**Line Bundle Extension (M., 2025)**

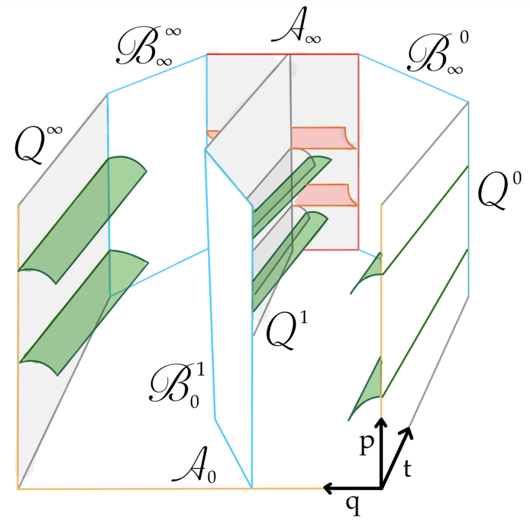
The trivial bundle $\mathcal{M} \times \mathbb{C}$ extends to

$$\mathscr{O}_{\widetilde{\mathcal{M}}}\left(A_0 + 2Q^1 + 2B_0^1 - B_\infty^0 - B_\infty^\infty\right).$$

Moreover, it is trivial when restricted to $Q^0$, $Q^\infty$ and $Q^1 \cup A_\infty$.



# Compactification $\overline{\mathfrak{Con}}_V^{\Theta}$

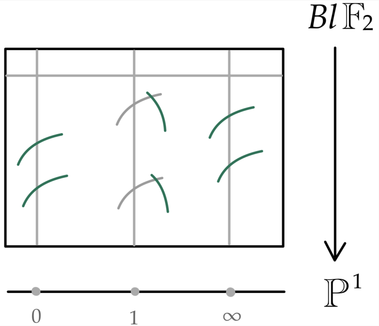


# Compactification $\overline{\mathfrak{Con}_V^{\Theta}}$

## Theorem

The compactified moduli space $\overline{\mathfrak{Con}_V^{\Theta}}$ comes with a fibration $\pi : \overline{\mathfrak{Con}_V^{\Theta}} \xrightarrow{t} \mathbb{P}^1$ such that:

- For any $t \in \mathbb{C}^*$, the fiber $\pi^{-1}(t)$ is an Okamoto space, that is a 8-blow-up of the second Hirzebruch surface $\mathbb{F}_2$.



# COMPACTIFICATION $\overline{\mathfrak{Con}_V^{\Theta}}$

**Theorem (M, 2025)**

The compactified moduli space $\overline{\mathfrak{Con}_V^{\Theta}}$ comes with a fibration $\pi : \overline{\mathfrak{Con}_V^{\Theta}} \xrightarrow{t} \mathbb{P}^1$ such that:

- The surface $\pi^{-1}(0)$ is given by $\mathcal{A}_0 \cup \mathcal{B}_0^1$. Both these components are isomorphic to $\mathbb{F}_1$.
- Let us denote by $\mathcal{F}_\infty^+$ and $\mathcal{F}_\infty^-$ the blow up of the two special sections in $\mathcal{A}_\infty$. The surface $\pi^{-1}(\infty)$ is given by $\mathcal{B}_0^\infty \cup \mathcal{B}_\infty^\infty \cup \mathcal{F}_\infty^+ \cup \mathcal{F}_\infty^-$. The surfaces $\mathcal{B}_0^\infty$ and $\mathcal{B}_\infty^\infty$ are isomorphic to $\mathbb{F}_1$.

# Painlevé V Foliation

# Painlevé V Equation

## The Equation (PV)

$$q''(t) = \left(\frac{1}{2q(t)} + \frac{1}{q(t)-1}\right) q'(t)^2 - \frac{1}{t} q(t)'$$

$$+ \frac{(q(t)-1)^2}{t^2}\left(\alpha q(t) + \frac{\beta}{q(t)}\right) + \gamma \frac{q(t)}{t} + \delta \frac{q(t)(q(t)+1)}{q(t)-1}$$

Where $\alpha, \beta, \gamma, \delta$ are parameters depending on $\Theta$.

# Painlevé V Equation

## The Hamiltonian Function

$$H^V := \frac{q(q-1)^2 p^2 - \Big(\kappa_0 (q-1)^2 + \kappa_1 q (q-1) - tq\Big) p + \kappa_\infty (q-1)}{t}$$

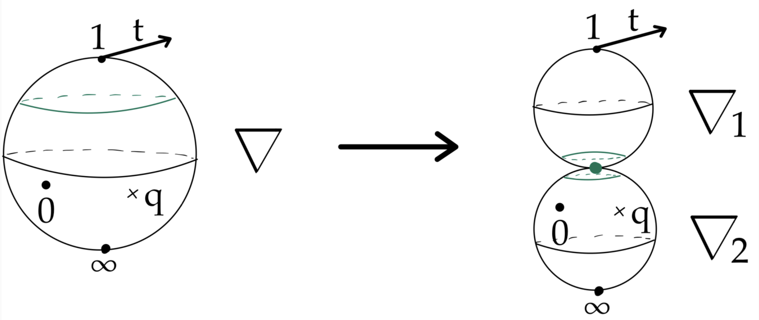
## The Hamiltonian System

$$\begin{cases} \dfrac{\partial H^V}{\partial p} = \dfrac{dq}{dt} \\[2ex] \dfrac{\partial H^V}{\partial q} = -\dfrac{dp}{dt} \end{cases}$$

# How to Find First Integrals

## Deformation of a PV connection

A PV connection splits into a connection over any smooth component. Both connections have a new pole at the node.



# How to Find First Integrals

**Proposition**

The residual spectral datum in the node is a first integral for the hamiltonian vector field in restriction to the respective boundary component.

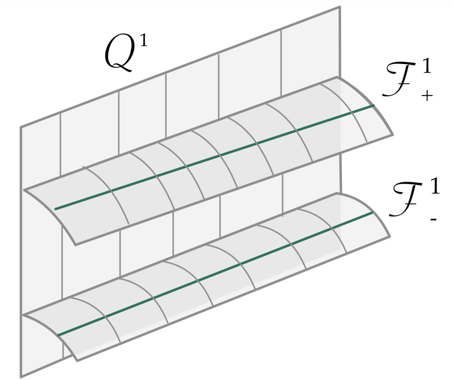
**Idea**

Since we are following an isomonodromic leaf, the monodromy along the loop $\gamma$ must be the same as the one of $\gamma_1$ and $\gamma_2$, corresponding to the monodromy of the new poles at the node. In particular the residual spectral datum in the node must be preserved during the deformation.

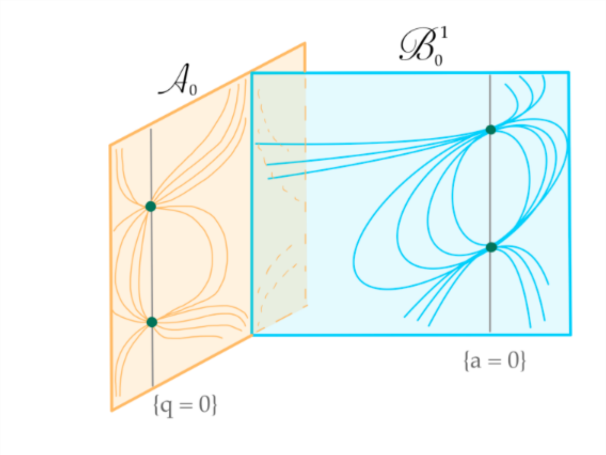
# Isomonodromic Foliation On the Okamoto Divisor

**Proposition**

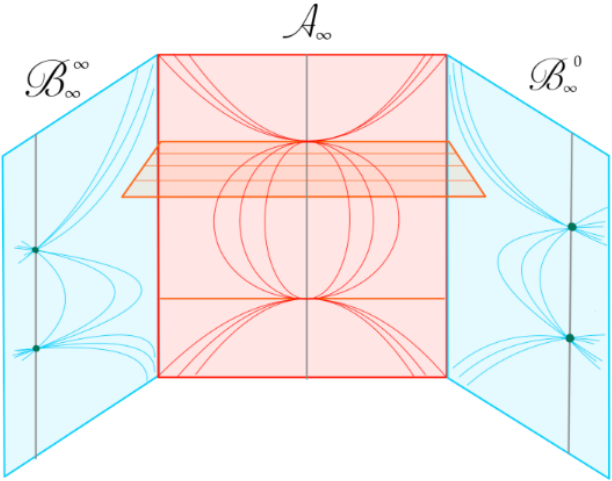
The isomonodromic foliation is stationary along the Okamoto divisor.



# First Integrals around $t = 0$



# First Integrals around $t = \infty$



# Thanks :)

Thanks for your attention !!