Compactification of the fifth Painlevé Foliation

Geometry and Dynamics Seminar, Cergy

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Meromorphic Connections

Meromorphic Connections on $\mathbb{P}^1$

Definition

A meromorphic connection (E, D, ∇) on $\mathbb{P}^1$ is the data of:

- a holomorphic vector bundle $E \to \mathbb{P}^1$,
- a effective divisor D of $\mathbb{P}^1$ called the polar divisor,
- a morphism $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathbb{P}^1}(D)$.

Leibniz Rule

The operator ∇ is required to satisfy

$$\nabla(f\sigma) = df \cdot \sigma + f\nabla\sigma$$

Warning

For this talk we always suppose $\mathrm{rk}(E) = 2$.

An example

Meromorphic connections on the trivial bundle

Let $E \cong \mathscr{O} \oplus \mathscr{O}$ be the trivial bundle. Any (E, D, ∇) is of the form

$$\nabla = d + \Omega, \quad \text{for} \quad \Omega \in \mathfrak{gl}_2\left(\Omega^1(D)\right).$$

The action on a global section is then the following:

$$\nabla Y = \nabla \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} + \begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

for some $\omega_{i,j} \in \Omega^1(D)$

What if E is not trivial ?

Remark (Birkhoff-Grothendieck Theorem)

Any rank 2 vector bundle E on $\mathbb{P}^1$ is of the form

$$E \cong \mathscr{O}(k_1) \oplus \mathscr{O}(k_2), \quad \text{for} \quad k_1, k_2 \in \mathbb{Z}.$$

Remark

The cover $\{U_0, U_\infty\}$ trivialize any vector bundle E on $\mathbb{P}^1$.

WHAT IF E IS NOT TRIVIAL ?

Gluing conditions

Any connection (E, D, ∇) on $\mathbb{P}^1$ is identified with the data

$$\begin{cases} d + \Omega_0 & \text{on } U_0 \\ d + \Omega_\infty & \text{on } U_\infty \end{cases} \quad \text{and cocycle} \quad g_{0,\infty}$$

The gluing condition on the overlap is then the following

$$\Omega_0 = g_{0,\infty}^{-1} \cdot \Omega_\infty \cdot g_{0,\infty} + g_{0,\infty}^{-1} \cdot dg_{0,\infty},$$

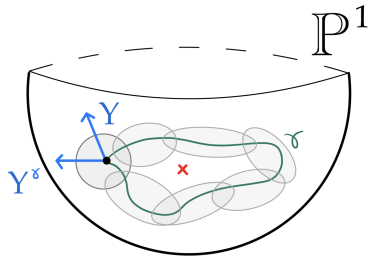
where $\{g_{0,\infty}\}$ is the cocycle of the bundle E.

Local Monodromy

Horizontal sections

A local section Y is said horizontal if $\nabla_{|U} Y = dY + \Omega_U Y = 0$

Monodromy



$$Y^\gamma = M_\gamma \cdot Y \quad \text{for} \quad M_\gamma \in \mathrm{GL}_2(\mathbb{C}).$$

Equivalence of Connections

Gauge equivalence of connections

We say that $(E, D, \nabla) \sim (E', D', \nabla')$ if there exists $\Phi \in \mathrm{Mor}(E, E')$ such that

$$\Phi^* \nabla' = \nabla.$$

Fact

$$(E, D, \nabla) \sim (E', D', \nabla') \quad \Longrightarrow \quad M_\nabla \sim M_{\nabla'}$$

Local equivalence

The connections matrices are locally related by:

$$\Omega_U = \Phi^{-1} \Omega'_U \Phi + \Phi^{-1} d\Phi$$

Painlevé V Connections

CONNEXIONS DU TYPE PV

Definition

Meromorphic connections (∇, E, D) such that

$$D^{\text{min}} = [0] + 2[1] + [\infty],$$

where D^{min} is the *minimal* polar divisor w.r.t. the gauge equivalence class of (E, D, ∇).

Consequence

In D^{min} only appear poles (equivalently)

- that cannot be eliminated nor reduced via a gauge transformation,
- with minimal Poincaré rank,
- with non-trivial monodromy.

Normal Form

Normal Form on $\mathscr{O} \oplus \mathscr{O}(2)$ (Diarra, Loray 2019)

$$\nabla_{|0} = d + \Omega_0 =$$

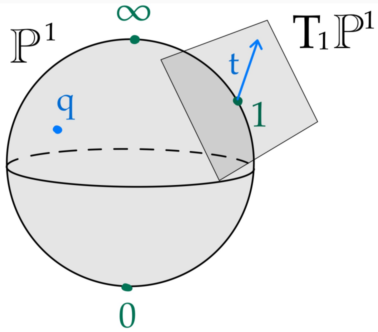
$$d + \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} 0 & -1 \\ 0 & \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & 1 \\ 0 & -\kappa_0 \end{pmatrix} \frac{dx}{x}$$

$$+ \begin{pmatrix} 0 & 0 \\ \kappa_\infty & 0 \end{pmatrix} x dx + \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ \hat{K} & 0 \end{pmatrix} dx,$$

Fixed Parameters

$$\Theta := \{\kappa_0, \kappa_1, \kappa_\infty\} \subseteq \mathbb{C}$$

Geometrical Interpretation of t and q



Action of $f \in \mathrm{Aut}(\mathbb{P}^1)$

Let $f \in \mathrm{Aut}(\mathbb{P}^1)$, then:

- $q \mapsto f(q)$,
- $t \mapsto Df(1) \cdot t$.

Rational Irregular Curves

Definition

A rational irregular curve is $(\mathbb{P}^1, D, J)$, where

- D is an effective divisor,
- J is a collection of jets.

Example

$(\mathbb{P}^1, D, J)$ with

- $D = [0] + 2[1] + [\infty] + [q]$
- $J = \{\, j^0(0),\ j^1(1) = (1, t),\ j^0(\infty),\ j^0(q)\,\}$.

Proposition

$[(\nabla, E, D)]$ "generic" of PV type $\iff$ a rational irregular curve and $p \in \mathbb{C}$.

Questions

Isomonodromic deformations

How can we deform the parameters (t, q, p) without changing the monodromy of the connection?
What happens when $t = 0$ or $t = \infty$?

Tools

- Moduli space of PV connections,
- Painlevé V equation,
- Isomonodromic foliation.

Moduli Space and Compactification

MODULI SPACE

Proposition

$[(\nabla, E, D)]$ "generic" of PV type $\iff$ a rational irregular curve and $p \in \mathbb{C}$.

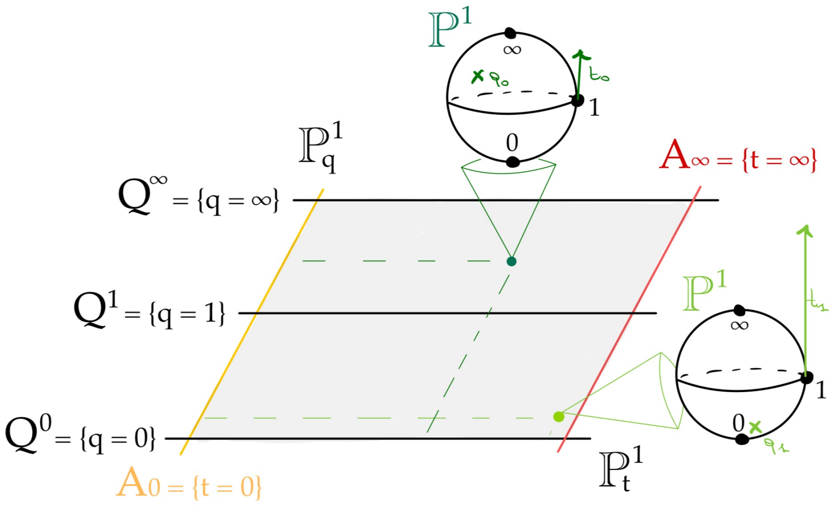
Moduli space of rational irregular curves $\mathcal{M}$

$$\mathcal{M} := \left\{ q \in \mathbb{P}^1 \setminus \{0, 1, \infty\};\ \ t \in T_1\mathbb{P}^1 \setminus \{0\} \right\}.$$

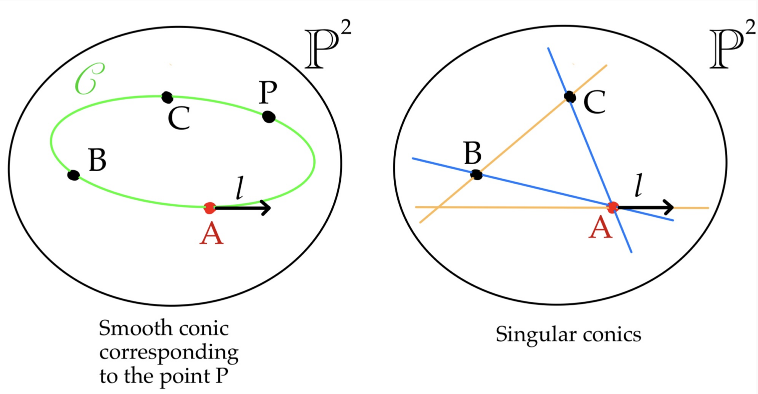
Moduli space of PV connections

$$\mathfrak{Con}_{\Theta}^{V} \supseteq \mathcal{M} \times \mathbb{C}$$

Moduli Space $\mathcal{M}$



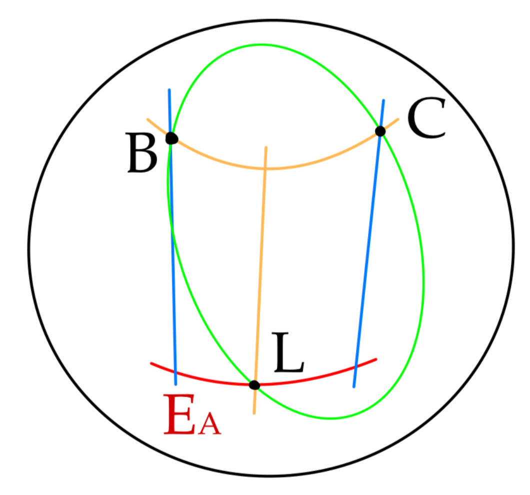
Moduli Space of Conics



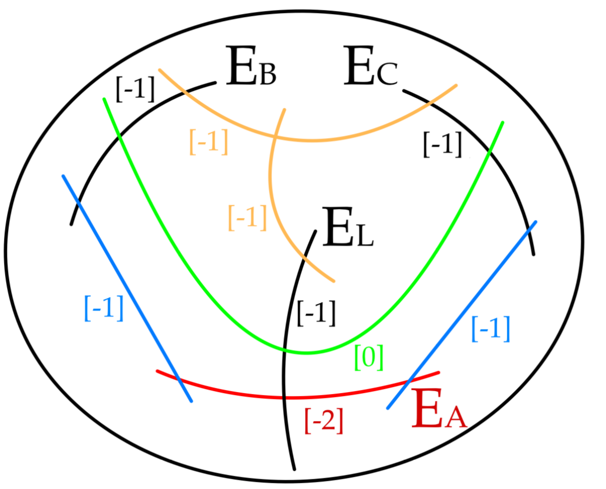
Moduli Space of Smooth Conics

$$\mathbb{P}^2 \setminus \left(\overline{AB}, \overline{AC}, \overline{BC}, l\right) \cong \mathcal{M}$$

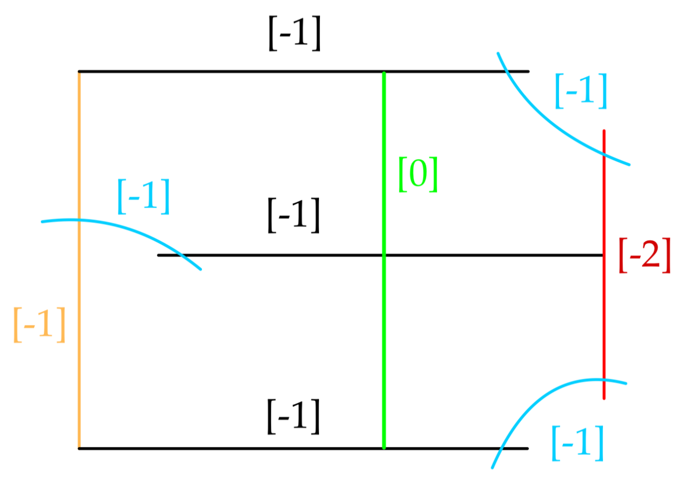
BLOW-UP OF A



Blow-up of B, C and L



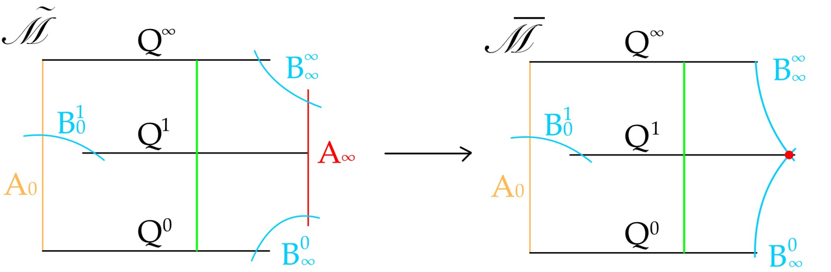
Compactification $\overline{\mathcal{M}}$



Compactification $\overline{\mathcal{M}}$

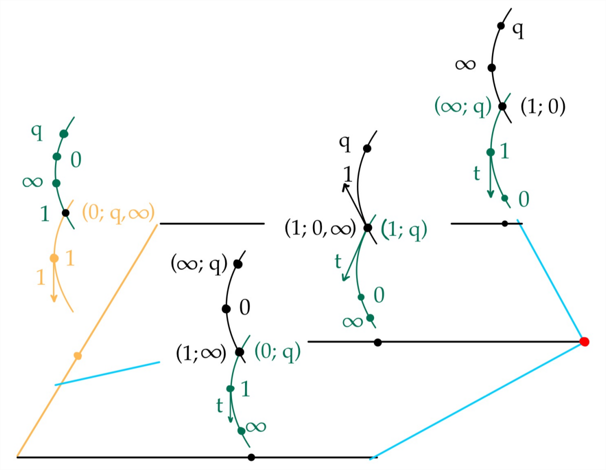
Compactification (M., 2025)

Following, *mutatis mutandis*, a similar idea that Kapranow used to compactify the spaces $\mathcal{M}_{0,n}$, we get:



where $\widetilde{\mathcal{M}}$ is the weak del Pezzo surface of degree five.

Universal Curve



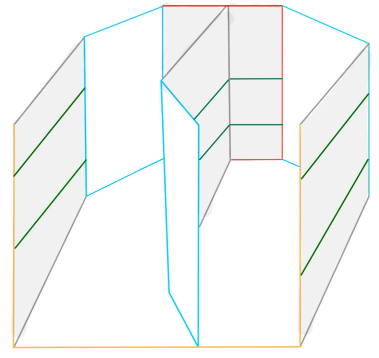
Compactification $\overline{\mathfrak{Con}_V^{\Theta}}$

Line Bundle Extension (M., 2025)

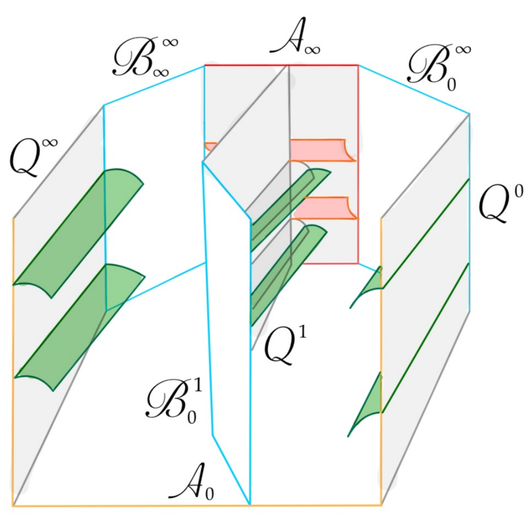
The trivial bundle $\mathcal{M} \times \mathbb{C}$ extends to

$$\mathscr{O}_{\widetilde{\mathcal{M}}}\left(A_0 + 2Q^1 + 2B_0^1 - B_\infty^0 - B_\infty^\infty\right).$$

Moreover, it is trivial when restricted to Q^0, Q^∞ and $Q^1 \cup A_\infty$.



Compactification $\overline{\mathfrak{Con}_V^{\Theta}}$



COMPACTIFICATION $\overline{\mathfrak{Con}_V^{\Theta}}$

Theorem (M, 2025)

The compactified moduli space $\overline{\mathfrak{Con}_V^{\Theta}}$ comes with a fibration $\pi\colon \overline{\mathfrak{Con}_V^{\Theta}} \xrightarrow{t} \mathbb{P}^1$ such that:

- For any $t \in \mathbb{C}^*$, the fiber $\pi^{-1}(t)$ is an Okamoto space, that is a 8-blow-up of the second Hirzebruch surface $\mathbb{F}_2$.
- The surface $\pi^{-1}(0)$ is given by $\mathcal{A}_0 \cup \mathcal{B}_0^1$. Both these components are isomorphic to $\mathbb{F}_1$.
- Let us denote by $\mathcal{F}_\infty^+$ and $\mathcal{F}_\infty^-$ the blow up of the two special sections in $\mathcal{A}_\infty$. The surface $\pi^{-1}(\infty)$ is given by $\mathcal{B}_0^\infty \cup \mathcal{B}_\infty^\infty \cup \mathcal{F}_\infty^+ \cup \mathcal{F}_\infty^-$. The surfaces $\mathcal{B}_0^\infty$ and $\mathcal{B}_\infty^\infty$ are isomorphic to $\mathbb{F}_1$.

Painlevé V Foliation

Painlevé V Equation

The Equation (PV)

$$q''(t) = \left(\frac{1}{2q(t)} + \frac{1}{q(t)-1}\right)q'(t)^2 - \frac{1}{t}q(t)'$$

$$+\frac{(q(t)-1)^2}{t^2}\left(\alpha q(t) + \frac{\beta}{q(t)}\right) + \gamma\frac{q(t)}{t} + \delta\frac{q(t)(q(t)+1)}{q(t)-1}$$

Where $\alpha, \beta, \gamma, \delta$ are parameters depending on Θ.

Painlevé V Equation

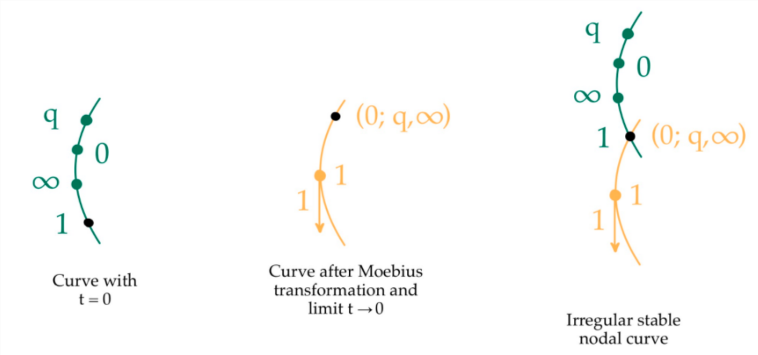
The Hamiltonian Function

$$H^V := \frac{q(q-1)^2 p^2 - \Big(\kappa_0 (q-1)^2 + \kappa_1 q (q-1) - tq\Big) p + \kappa_\infty (q-1)}{t}$$

The Hamiltonian System

$$\begin{cases} \dfrac{\partial H^V}{\partial p} = \dfrac{dq}{dt} \\[2ex] \dfrac{\partial H^V}{\partial q} = -\dfrac{dp}{dt} \end{cases}$$

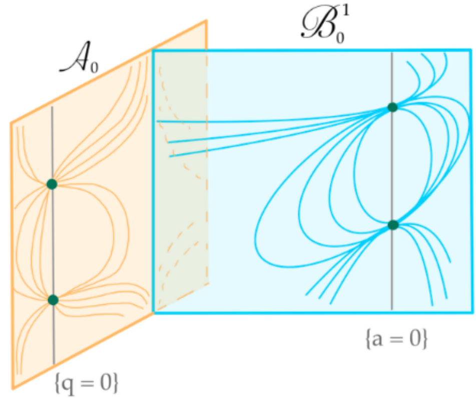
How to Find First Integrals



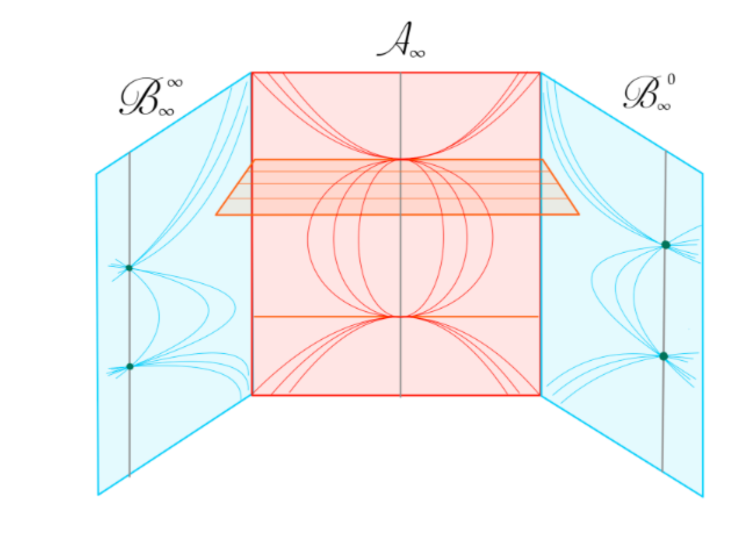
Proposition

The residual spectral datum in the node is a first integral for the hamiltonian vector field in restricition to the respective boundary component.

First Integrals around $t = 0$



First Integrals around $t = \infty$



Thanks :)

Thanks for your attention !!