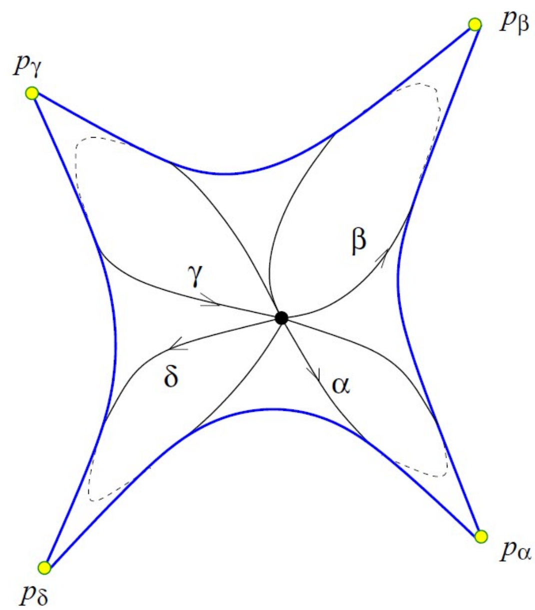
Université de Rennes 1 · UFR Mathématiques

Master 2 Mathématiques Fondamentales

Character Varieties, the 4-Punctured Sphere, and Symplectic Structures

Mattia Morbello

Encadrant : Frank Loray



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Introduction

The main goal of this work is to point out the geometry of the character varieties, passing through the example of $\mathbb{S}^2_4$, the 4-punctured sphere.
The interest of character varieties, besides their rich geometry, relies in the Riemann-Hilbert correspondence, that states, in some cases, a symplectomorphism between the character variety of a closed (i.e. non punctured) surface and the variety of flat connections of a vector bundle modulo the gauge action. In order to become more familiar with this object, in the first section we will present the most general theory, introducing all the different actors we will meet and study in the sequel. Maybe the reader will get a bit lost as in the Dante's *gloomy wood*, but after some struggle all the elements will find their place in order to create a logical scheme in the reader's mind.
The second section is devoted to see the special and interesting example of the 4-punctured sphere. Here we can describe in details the action of the Mapping Class Group and we will start to see how some objects we have defined start to play one with the other in this very concrete environment. In the end of this section we provide a diagram, that should be the compass to navigate this sea.
While the second section aims to study the dynamics of the character variety, the last one is way more geometric. We leave the example of the 4-punctured sphere that helped us to get familiar with the first definitions: in this case the symplectic geometry we want to study becomes harder and hostile. We then move to a closed surface, that allows the definition of a symplectic structure on its character variety. In this third section we will build the symplectic form in details, seizing the opportunity to introduce the Riemann-Hilbert correspondence.

I want to express my gratitude to my advisor, that patienly helped me integrating in this world (mathematical and not), completely new for me. Many thanks also go to Matilde, Alice, Sergio, Remi, Axel and Marc who welcomed me making my work easier and funnier.

1 Preliminaries

In this section the main ingredients of our research are introduced. At a first glance, they will seem not much connected to each other, but in the second section a beautiful link between them will rise up under the form of an action of the mapping class group on the character variety.

1.1 Punctured Surfaces and Mapping Class Group

We consider a (topological) compact connected orientable surface S and we subtract to it a finite number of points, we will then say that S has been punctured. By the classification theorem of surfaces, S is determined (up to homeomorphism) by its genus and by the number of punctures.

Remark. We briefly open and close a parenthesis: in some literature punctured surfaces are considered as surfaces with boundary, where the boundary components arise in "enlarging in a closed way the punctures". For example a single punctured sphere can be considered as a closed disk and a double punctured sphere can be considered as a closed annulus. This will lead to a slightly different approach when we will talk about orientation preserving homeomorphisms, because in this case boundary component must be fixed. We will not go through this way and for us a punctured surface will be just a surface minus a finite set of points.

Therefore, up to homeomorphisms, it suffices to give two integer numbers in order to classify all possible punctured surfaces.

Definition. We will denote by $S_{g,b}$ the punctured surface of genus g and with b punctures.

Example 1. The 4-punctured sphere $\mathbb{S}^2_4$ is the punctured surface $S_{0,4}$ and it is, for example, homeomorphic to $\mathbb{CP}^1 \setminus \{0, 1, t, \infty\}$ for $t \neq 0, 1$ a complex number.

It is clear that subtracting a finite number of points will strongly change the topology of the surface. In particular a lot of non trivial loops will arise around the punctures and then it is natural to consider the fundamental groups of punctured surfaces.

Definition. A *surface group* Γ is a group that can be realised as the fundamental group of a punctured surface $S_{g,b}$.

In particular any surface group can be presented as :

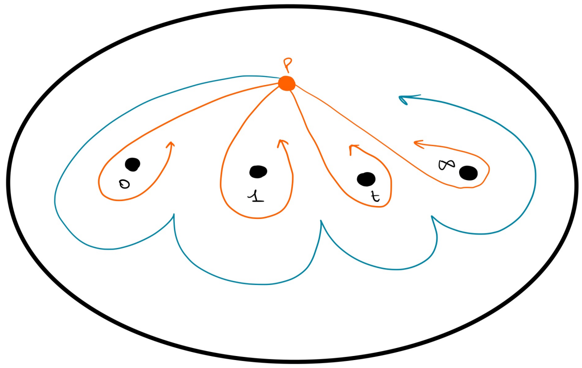
$$\Gamma := \pi_1(S_{g,b}) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_b \;\middle|\; \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{j=1}^{b} c_j = 1 \right\rangle,$$

where the c_j are the loops around the punctures.

Example 2. Let us consider $\mathbb{S}^2_4 = S_{0,4} = \mathbb{CP}^1 \setminus \{0, 1, t, \infty\}$. In this case we have (since $g = 0$) that

$$\Gamma = \left\langle c_1, c_2, c_3, c_4 \;\middle|\; \prod_{j=1}^{4} c_j = 1 \right\rangle$$

and we can easily explain the relation $\prod_{j=1}^{4} c_j = 1$ via the following picture.



The surfaces groups are pairwise non isomorphic and it can be shown that any group Γ presented in this way can be realised as the fundamental group of a punctured surface (eventually with no punctures if $b = 0$). We refer to [7] for any detail.

For the sequel we fix an orientation of a punctured surface S.

Definition. The *extended mapping class group* of S, denoted by $\mathrm{MCG}^*(S)$, is the set of isotopy classes of homeomorphisms of S, i.e. $\mathrm{MCG}^*(S) := \mathrm{Homeo}(S)/\!\sim_{\text{isotopy}}$.

In the literature the mapping class group, denoted by $\mathrm{MCG}(S)$, is defined as the set of isotopy classes of *orientation preserving* homeomorphisms, that is an index 2 subgroup of $\mathrm{MCG}^*(S)$. In the following sections we will see why for us is more interesting the "extended" version. Since we work with punctured surfaces (and not with surfaces with boundary, as specified in the previous remark), an another natural choice we do, not always shared by all the literature, is that we allow permutations of the punctures.

In order to better understand the extended mapping class group, we recall the following definition:

Definition. Let f, g be two homeomorphisms of S. An *isotopy* between them is a continuous map $H\colon S \times [0,1] \to S$ such that

- $H(\cdot, 0) = f$,
- $H(\cdot, 1) = g$,
- $H(\cdot, t)$ is a homeomorphism of S.

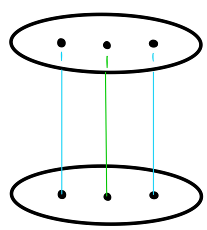
Remark. The first two conditions say that H is an homotopy between f and g, but the third conditions assure the invertibility of any intermediate function (and not only the continuity as for the homotopy).

Thanks to this definition and by the fact that we can give $\mathrm{MCG}^*(S)$ the compact-open topology, we can consider isotopies as continuous paths in $\mathrm{Homeo}(S)$ and hence it holds that $\mathrm{MCG}^*(S) \cong \pi_0\big(\mathrm{Homeo}(S)\big)$.

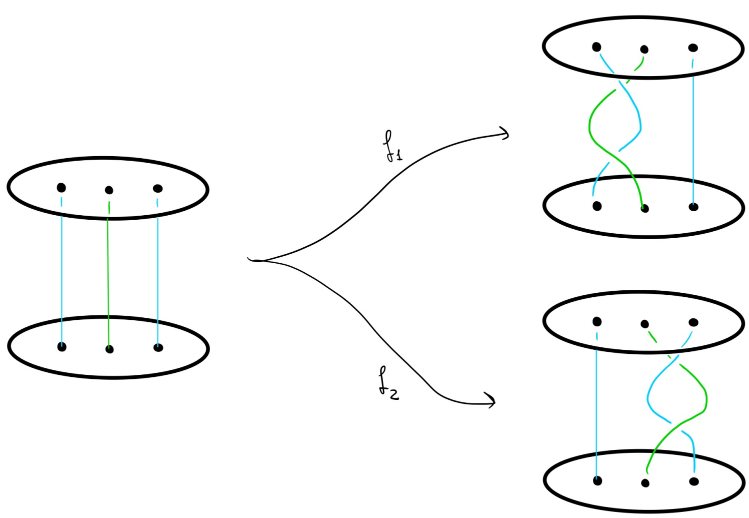
To make a recap we have the following diagram

$$\begin{tikzcd}
& \mathrm{MCG}(S) \arrow[dr, hook] & \\
\mathrm{MCG}_0(S) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{MCG}^*(S) \\
& \mathrm{MCG}_0^*(S) \arrow[ur, hook] &
\end{tikzcd}$$

Where on the left there is the subgroup preserving the punctures and the orientation, on the top the subgroup preserving the orientation, on the bottom the subgroup preserving the punctures and on the right, the extended mapping class group that can reverse the orientation and can permute the punctures.
We conclude this section with a more "practical" interpretation of a subgroup of $\mathrm{MCG}^*(S)$ for $S = \mathbb{S}_b^2$, the b-punctured sphere. The group $\mathrm{MCG}_0(\mathbb{S}_b^2)$ can in fact be visualised in terms of braid groups. We will not go through the details of the definition since we are more interested in a geometric representation of the braid groups than in its formal description. In particular we will consider the braid group of 3 strands. We can build it by considering two copies of a 3-punctured closed disk (that by the previous remark we recall we can see as a 4-punctured sphere) and by imaging that each puncture of the first disk is connected to its corresponding point in the other disk, as shown in the figure.



The braid group of 3 strands is then the group given by all possible braids we can create "twisting the points" without changing their order. Inside the braid group of 3 strands we can consider the free group generated by the two elements shown in the following picture.



In particular it represents exactly what happens when the $\mathrm{MCG}_0(\mathbb{S}^2_4)$ acts on the 4-punctured sphere: in fact it is obliged to fix the four punctures and the orientation (and then preserves the three punctures and the boundary of the disk) and the other points are "twisted" around the punctures. It is interesting to remark that $\mathrm{MCG}_0(\mathbb{S}^2_4)$ then is isomorphic to the free group $\mathbb{F}_2$.

1.2 Representation Variety

Let Γ be a surface group and G a Lie group. A representation of Γ onto G is a group morphism $\rho\colon \Gamma \to G$. What is said in this section is true for any G reductive, but the reader should always think at G as $\mathrm{SL}(2,\mathbb{C})$ because it is the group we will adopt in the following sections.

Definition. The *representation variety* associated to a surface group Γ is the set $\mathrm{Hom}(\Gamma, G)$ composed by all the representations $\rho\colon \Gamma \to G$.

Being G a Lie group, it is also a topological space, and we can give $\mathrm{Hom}(\Gamma, G)$ the compact-open topology induced by the total space G^Γ in which $\mathrm{Hom}(\Gamma, G)$ lives as a subspace. Moreover, since group morphisms in some sense preserve the relations of Γ, we can see $\mathrm{Hom}(\Gamma, G)$ as a subset of G^n, where $n = 2g + b$. In fact we have the following

Proposition 1.1. *The set*

$$X(\Gamma, G) := \{\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g), \rho(c_1), \dots, \rho(c_b) \mid \rho \in \mathrm{Hom}(\Gamma, G)\} \subseteq G^n$$

is an analytic subvariety of G^n homeomorphic to $\mathrm{Hom}(\Gamma, G)$*. Moreover its analytic structure does not depend by the choice of the generators of* Γ*.*

Proof. See [7], Lemmata 1.2.3 and 1.2.4. □

Remark. If in Γ there are no relations, then $\mathrm{Hom}(\Gamma, G) \cong X(\Gamma, G) = G^n$.

We are interested in studying group actions on the representation variety. Being G and Γ two groups, the first idea is to consider the action of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\Gamma)$ on it: in fact

- $\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{Hom}(\Gamma, G)$ is a right action by pre-composition: if $\varphi \in \mathrm{Aut}(\Gamma)$,

$$\varphi(\rho)(\gamma) := \rho(\varphi^{-1}(\gamma)).$$

- $\mathrm{Aut}(G) \curvearrowright \mathrm{Hom}(\Gamma, G)$ is a left action by post-composition: if $f \in \mathrm{Aut}(G)$,

$$f(\rho)(\gamma) := f(\rho(\gamma)).$$

As we will see, both have a meaning for us, but for the moment we study the $\mathrm{Aut}(G)$ action. In fact inside this automorphisms group there is the normal subgroup of inner automorphisms, $Inn(G) := \{C_g\colon h \mapsto ghg^{-1} \mid g \in G\}$. This group assume a crucial interest if we recall that two representations $\rho, \rho' \in \mathrm{Hom}(\Gamma, G)$ are said equivalent (and we denote it by $\rho \sim \rho'$) if and only if there exist an element $g \in G$ such that for any $\gamma \in \Gamma$ it holds that $\rho(\gamma) = g\rho'(\gamma)g^{-1}$. Let us fix $G = \mathrm{SL}(2,\mathbb{C})$. This definition makes sense because we say that, $\rho(\gamma), \rho'(\gamma) \in \mathrm{SL}(2,\mathbb{C})$ acts in the same way on the vector space $\mathbb{C}^2$ up to a basis change, i.e. for any $\gamma \in \Gamma$ the following diagram commutes, where $g \in \mathrm{SL}(2,\mathbb{C})$ is considered as a basis change of $\mathbb{C}^2$.

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "\rho(\gamma)"] \arrow[d, "g"] & \mathbb{C}^2 \arrow[d, "g"] \\ \mathbb{C}^2 \arrow[r, "\rho'(\gamma)"'] & \mathbb{C}^2 \end{tikzcd}$$

In particular we would really like, in order to study the representation variety, to consider the quotient

$$\mathrm{Hom}(\Gamma, G)/Inn(G).$$

But, as always happens in the best stories, not everything goes as expected. In fact in general the $Inn(G)$-action is neither free (the trivial representation is always a fixed point) nor proper and then the quotient is a very ugly topological space (for instance, it is non Hausdorff).

Remark. In the section 1.5 we will better explain the behaviour of this action.

The next section is devoted to try to solve the problem.

1.3 The Character Variety

The main problem of the action of $Inn(G)$ is that the closure of an orbit may intersect the closure of a different orbit, making the quotient non Hausdorff. Let us explain this with two detailed examples. We will also seize the opportunity for giving some details (in the easier context of the example) we skipped before.

Example 3. Let consider $\Gamma \cong \mathbb{Z}$ and $G = \mathrm{SL}(2, \mathbb{C})$. We stress that $\mathbb{Z} \cong \langle a \rangle$ is not a surface group, but we will anyway consider it because it is really easy to handle. First of all we can prove that

Fact. The representation variety $\mathrm{Hom}(\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}))$ is isomorphic to $\mathrm{SL}(2, \mathbb{C})$.

Proof. Let us consider the map

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}, \mathrm{SL}(2, \mathbb{C}) & \to & \mathrm{SL}(2, \mathbb{C}) \\ \rho & \mapsto & \rho(1). \end{array}$$

It is injective because if $\rho(1) = Id$ then $\rho(-1) = Id$ and this force ρ to be the trivial representation.
It is surjective because, if we consider $g \in G$, the map $\rho_g \colon n \mapsto g^n$ is in $\mathrm{Hom}(\mathbb{Z}, G)$ and is such that $\rho_g(1) = g$. □

Therefore we can consider the quotient

$$\mathrm{SL}(2, \mathbb{C})/Inn(\mathrm{SL}(2, \mathbb{C}))$$

and then study the conjugation classes in $\mathrm{SL}(2, \mathbb{C})$. We have two different situations:

- Diagonalisable: $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \sim \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$
- Non diagonalisable: $A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \nsim \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} =: B$

Therefore

$$\mathrm{SL}(2,\mathbb{C})/Inn(\mathrm{SL}(2,\mathbb{C})) \cong \mathbb{C}^*/(\lambda \sim \lambda^{-1}) \cup \{[A],[B]\}$$

and we have that

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix} \xrightarrow{\lambda\to 0} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

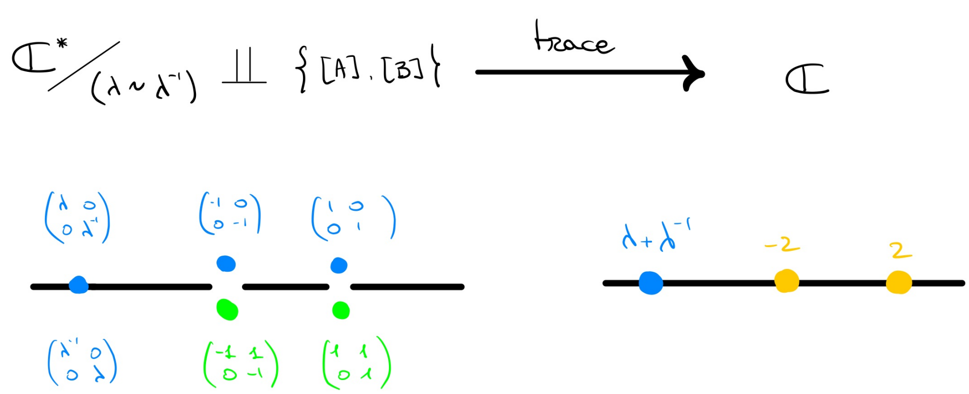
that means $\overline{O(A)} \cap \overline{O(Id)} \neq \emptyset$ and then $\mathbb{C}^*/(\lambda \sim \lambda^{-1}) \cup \{[A],[B]\}$ is not Hausdorff. We want to find a way to identify these orbits. The first ideas to notice that when $\lambda = \pm 1$, we have $\lambda + \lambda^{-1} = \pm 2$, that means that the trace is not able to distinguish $[A]$ from $[Id]$ and $[B]$ from $[-Id]$. We can summarise this informations in the following diagram:

$$\begin{array}{ccccccc}
\mathrm{SL}(2,\mathbb{C}) & \xrightarrow{proj} & \mathrm{SL}(2,\mathbb{C})/Inn(\mathrm{SL}(2,\mathbb{C})) & \xrightarrow{\cong} & \mathbb{C}^*/(\lambda \sim \lambda^{-1}) \cup \{[A],[B]\} & \longrightarrow & \mathbb{C} \\
g & \longmapsto & [g] & \longrightarrow & \begin{cases}\lambda\\ A\\ B\end{cases} & \longrightarrow & \begin{cases}\lambda+\lambda^{-1}\\ 2\\ -2\end{cases}
\end{array}$$

We can use this fact in order to identify these orbits and solve the problem, i.e. considering

$$\mathrm{Hom}(\Gamma, G)//Inn(G) := \left(\mathbb{C}^*/(\lambda \sim \lambda^{-1}) \cup \{[A],[B]\}\right)\Big/ Tr \ \cong \mathbb{C}.$$

To conclude we give a visual representation of this example.



Example 4. Let $\Gamma = \mathbb{F}_2 = \langle a, b\rangle$ and $G = \mathrm{SL}(2,\mathbb{C})$. Since in Γ there are no relations, we have that $\mathrm{Hom}(\Gamma, G) \cong \mathrm{SL}(2,\mathbb{C})^2$, that means that any representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is completely determined by $(\rho(a), \rho(b)) \in \mathrm{SL}(2,\mathbb{C})^2$. Let us set

$$\rho_1(a) = A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ \rho_1(b) = B = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \rho_2(a) = \rho_2(b) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

i.e. ρ_2 being the trivial representation.

We hence have that the quotient is $\mathrm{SL}(2,\mathbb{C})^2/Inn(\mathrm{SL}(2,\mathbb{C}))$ and the orbits of ρ_1 and ρ_2 are of the form:

- $O(\rho_1) = \{(gAg^{-1}, gIdg^{-1}) \mid g \in \mathrm{SL}(2,\mathbb{C})\} = \{(gAg^{-1}, Id) \mid g \in \mathrm{SL}(2,\mathbb{C})\} \subseteq \mathrm{SL}(2,\mathbb{C})^2$.
- $O(\rho_2) = \{(gIdg^{-1}, gIdg^{-1}) \mid g \in \mathrm{SL}(2,\mathbb{C})\} = \{(Id, Id)\} = \{\rho_2\} \subseteq \mathrm{SL}(2,\mathbb{C})^2$.

As we have seen in the first example, then we have that $\rho_2 \in \overline{O(\rho_1)} \setminus O(\rho_1)$ and $O(\rho_2) = \{\rho_2\}$. That means that the two different orbits cannot be separated by disjoint open sets in the topological quotient, that thus is not Hausdorff.

Remark. While $\mathbb{Z}$ is not a surface group, the free group of two generators $\mathbb{F}_2$ is isomorphic to $\pi_1(\mathbb{S}^2_3)$.

Therefore a first rude idea is to glue together these problematic orbits in the quotient, but how can we do it in a suitable way? There are a lot of answers for this question, here we will try to give the main idea coming from GIT quotient, characters and trace functions.

Definition. Let $\gamma \in \Gamma$, the *trace function* associated to γ is

$$\begin{aligned} Tr_\gamma \colon \mathrm{Hom}(\Gamma, G) &\to \mathbb{C} \\ \rho &\mapsto \mathrm{Tr}(\rho(\gamma)). \end{aligned}$$

Remark. We stress the fact that trace function is invariant under the action of $Inn(G)$ because of its symmetry property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

This kind of functions (when G has suitable properties, like in our case) generate all the algebra of invariant functions $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. The idea coming directly from the algebraic geometry of schemes is the following: since what we are looking for should have as coordinate ring exactly $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$, we can set

$$\mathrm{Hom}(\Gamma, G)//Inn(G) := spec\Big(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G\Big),$$

where the double quotient stresses the difference between this GIT quotient and standard topological one. If we are dealing with a Lie group whose $Inn(G)$-invariant functions are generated by traces, we can give this double quotient a very concrete interpretation coming from the characters.

Remark. For $\mathrm{SL}(2,\mathbb{C})$ is easy to see that it is verified: it is a standard fact that invariant functions are generated by the coefficients of the characteristic polynomial. In particular for $\mathrm{SL}(2,\mathbb{C})$ this coefficient are just the trace and the determinant, that is constant.

Definition. The *character* of a representation ρ is the function

$$\begin{aligned} \chi_\rho \colon \Gamma &\to \mathbb{C} \\ \gamma &\mapsto \mathrm{Tr}(\rho(\gamma)). \end{aligned}$$

In particular it holds that $\chi_\rho(\gamma) = Tr_\gamma(\rho)$.
We denote by $\chi(\Gamma, G)$ the set of all characters and we have a natural projection $\mathrm{Hom}(\Gamma, G) \to \chi(\Gamma, G)$ that factors through the quotient $\mathrm{Hom}(\Gamma, G)/Inn(G)$:

$$\begin{tikzcd} \mathrm{Hom}(\Gamma, G) \arrow[rr, "\rho\mapsto\chi_\rho"] \arrow[d, "\rho\mapsto[\rho]"'] & & \chi(\Gamma, G) \\ \mathrm{Hom}(\Gamma, G)/Inn(G) \arrow[urr, "{[\rho]\mapsto\chi_\rho}"'] & & \end{tikzcd}$$

It can be proven that $\chi(\Gamma, G)$ is an algebraic variety (in particular an Hausdorff space) that is isomorphic to $\mathrm{Hom}(\Gamma, G)//Inn(G)$. Roughly speaking in the standard topological quotient $\mathrm{Hom}(\Gamma, G)/Inn(G)$ we had some troubles because of intersecting adherence of orbits and what we did is actually make an another quotient identifying the orbits that induce the same character, i.e. the orbits with the same trace.

1.4 The Zarisky Tangent Space

An important tool that we can use in order to study character varieties of closed (i.e. non punctures) surfaces is the symplectic geometry. In the suite of this work we will then focus on closed surface groups. We will need to go through the differential geometry of the representation variety, studying its tangent space in order to define differential forms on it. We will firstly do it straightforward, and then, after some preliminaries, we will give to this tangent space a more abstract description via the group cohomology.

Interlude: Group Cohomology

Group cohomology is a huge and very subtle argument. The intent of this short guide is to introduce the reader to the essential definitions that should lead to understand what follows. In particular we will use the definition via bar resolution and, since we will only need the first cohomology group, we will especially focus on it.
We recall that given a group Γ (for instance it will be our fundamental group), a Γ-module M is an abelian group on which Γ acts compatibly with the abelian group structure on M.
As usual, in order to point out a cohomology we need a cochain complex $\{C^n, \delta\}$, where (with the usual abuse of notation) $\delta\colon C^n \to C^{n+1}$ such that $\delta^2 = 0$. Then we set:

- The cochain sets:
$$C^n(\Gamma, M) := \{ \text{ functions } \ f : \Gamma^n \to M\}$$
- The boundary maps $\delta\colon C^n(\Gamma, M) \to C^{n+1}(\Gamma, M)$ given by
$$\delta f(\gamma_1, \dots, \gamma_{n+1}) :=$$
$$:= \gamma_1 f(\gamma_2, \dots, \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_{n+1}) + (-1)^{n+1} f(\gamma_1, \dots, \gamma_n)$$

For the sequel we will focus on the cases $n = 0, 1$, we will see how the boundary morphism looks like in this case and we will introduce cocycles, co boundaries and the first cohomology group. In particular, we recall that a 0-cochain is simply an element $v \in M$.

$$\begin{aligned} n &= 0 \qquad && \delta v(\gamma) = \gamma \cdot v - v \\ n &= 1 \qquad && \delta f(\gamma_1, \gamma_2) = \gamma_1 \cdot f(\gamma_2) - f(\gamma_1\gamma_2) + f(\gamma_1) \end{aligned}$$

where $\gamma \cdot v$ is the action of Γ defined on M.
In order to define the first cohomology groups we have to find out the group of 1-cocycles and 1-

coboundaries defined respectively as the kernel and the image of δ.

$$Z^1(\Gamma, M) = \ker(\delta_{|C^1}) = \{f \in C^1 \mid f(\gamma_1\gamma_2) = \gamma_1 \cdot f(\gamma_2) + f(\gamma_1)\}$$
$$B^1(\Gamma, M) = im(\delta_{|C^0}) = \{f \in C^1 \mid f(\gamma) = \gamma \cdot v - v, \text{ for a } v \in M\}.$$

Remark. By a direct computation we get that $B^1(\Gamma, M) \subseteq Z^1(\Gamma, M)$. Let $f_v = \delta v \in B^1(\Gamma, M)$.

$$\delta(f_v)(\gamma_1, \gamma_2) = \gamma_1 f_v(\gamma_2) - f_v(\gamma_1\gamma_2) + f_v(\gamma_1) = (\gamma_1\gamma_2 \cdot v - \gamma_1 \cdot v) - (\gamma_1\gamma_2 \cdot v - v) + (\gamma_1 \cdot v - v) = 0.$$

Finally we define the first cohomology group as

$$H^1(\Gamma, M) := \frac{Z^1(\Gamma, M)}{B^1(\Gamma, M)}.$$

Remark. What we gave is the simplicial picture of group cohomology. Anyway there is a more geometric interpretation of it in terms of classifying spaces. A classifying space for Γ is a topological space S with fundamental group isomorphic to Γ and with contractible universal covering. In this case we can prove (or take as definition) that

$$H^*(\Gamma, \mathbb{C}) \cong H^*_{DR}(S, \mathbb{C}).$$

We stress that we are dealing with compact closed complex surfaces of genus greater than 0, and then, by the uniformisation theorem, they are covered by the hyperbolic plane that is actually contractible.

Remark. Using the Riemann-Hilbert correspondence, as we will see in the last section, we can say more than this. In fact if moreover S is a complex surface it holds that

$$H^*(\Gamma, \mathfrak{g}_\rho) \cong H^*(S, E_\rho),$$

where $\mathfrak{g}_\rho$ is the Γ-module built via Ad_ρ and E_ρ is the flat vector bundle associated to the principal bundle $(\tilde{S} \times G)/\Gamma \to S$.

Direct Construction of the Tangent Space

Tanks to Proposition 1.1, we know that $\mathrm{Hom}(\Gamma, G)$ inherits an analytic structure from G^n: in fact we recall that it is the subspace defined by the equations

$$\rho(\alpha\beta)\rho(\alpha)^{-1}\rho(\beta)^{-1} = 1 \quad \text{for } \alpha, \beta \text{ generators of } \Gamma$$

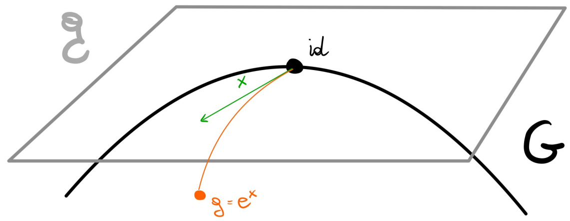
that states that ρ is not only a function $\Gamma \to G$, but it is actually a group morphism. We will then compute its tangent spaces at a smooth point of the quotient $\mathrm{Hom}(\Gamma, G)//Inn(G)$.

Definition (1)**.** Let $X \subseteq \mathbb{C}^n$ be an analytic variety defined as the zero locus of some functions $f_1, \dots, f_m : \mathbb{C}^n \to \mathbb{C}$. The *Zarisky tangent space* at $x \in X$ is the kernel of the $m \times n$ Jacobi matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j}.$$

Definition (2)**.** Let $X \subseteq \mathbb{C}^n$ be an analytic variety defined as the zero locus of some functions $f_1, \dots, f_m : \mathbb{C}^n \to \mathbb{C}$. The *Zarisky tangent space* at $x \in X$ consists of all tangent vectors $x'(0)$ tangent to a smooth path $x(t)$ inside $\mathbb{C}^n$ with $x(0) = x$ and that lies on the variety, i.e. such that $f_i(x(t)) = 0$.

We recall that the tangent space at the identity of a Lie group G is its Lie algebra $\mathfrak{g}$ and that we have the exponential map $\exp\colon \mathfrak{g} \to G$ that allows us to produce analytic paths $t \mapsto e^{tX}$, for $X \in \mathfrak{g}$. If $g = e^X$ is an element of G, the path e^{tX} connects g and id and has as tangent vector in the identity the element $X \in \mathfrak{g}$.



We remark that if $h \in G$, then the path $e^{tX}h$ has tangent vector in h equal to X.

Remark. Locally and at the first order, any path passing through h is of the form $e^{tX}h$ for an element $X \in \mathfrak{g}$. This fact turns out to be crucial for us: in fact we are interested to compute the tangent space of $\operatorname{Hom}(\Gamma, G)$ at a point ρ and to do so, we will use the definition (2) that involves curves first derivative, and hence we are interested in (local) curves passing through ρ up to the first order.

Let $\rho_t := \rho(t)$ be a path of representations such that $\rho_0 = \rho$. For each $\gamma \in \Gamma$ there exists an element $c(\gamma) \in \mathfrak{g}$ such that

$$\rho_t(\gamma) = e^{tc(\gamma)}\rho_0(\gamma) + o(t)$$

at first order. We can now use the equations defining $\operatorname{Hom}(\Gamma, G)$ on ρ_t, that gives

$$\rho_t(\gamma_1\gamma_2) = \rho_t(\gamma_1)\rho_t(\gamma_2)$$

which at first order gives

$$e^{tc(\gamma_1\gamma_2)}\rho(\gamma_1\gamma_2) = e^{tc(\gamma_1)}\rho(\gamma_1)e^{tc(\gamma_2)}\rho(\gamma_2) + o(t).$$

We can then use Definition (2) and differentiate this expression in $t = 0$:

$$c(\gamma_1\gamma_2)\rho(\gamma_1\gamma_2) = c(\gamma_1)\rho(\gamma_1)\rho(\gamma_2) + \rho(\gamma_1)c(\gamma_2)\rho(\gamma_2).$$

Then by using the equation and multiplying on the right by $\rho(\gamma_2)^{-1}\rho(\gamma_1)^{-1}$ we finally get

$$c(\gamma_1\gamma_2) = c(\gamma_1) + \rho(\gamma_1)c(\gamma_2)\rho(\gamma_1)^{-1} = c(\gamma_1) + \operatorname{Ad}_{\rho(\gamma_1)} c(\gamma_2)$$

Proposition 1.2. *The Zarisky tangent space of* $\operatorname{Hom}(\Gamma, G)$ *at a point* ρ *is given by*

$$T_\rho \operatorname{Hom}(\Gamma, G) = \{c \in \mathfrak{g}^\Gamma \mid c(\gamma_1\gamma_2) = \operatorname{Ad}_{\rho(\gamma_1)} c(\gamma_2) + c(\gamma_1)\}.$$

Now that we have this powerful result, we should focus on what happens at the tangent space of the quotient $\chi(\Gamma, G) = \operatorname{Hom}(\Gamma, G)//Inn(G)$.
If the action were proper and free, we will just say that the tangent space of the orbit space is the quotient of the tangent space of the variety by the tangent space of the orbit: in formulae

$$T_\rho\chi(\Gamma, G) = T_\rho \operatorname{Hom}(\Gamma, G)/T_\rho\mathcal{O}(\rho).$$

We know that it is not the case, but in the next section we will talk about smooth points and we will see that on the Zarisky open set of irreducible representation the action is proper and locally free and we can then consider this tangent space. So, let us compute the tangent space of the orbit. We will proceed as before: let $\rho_t := \rho(t)$ a path such that $\rho_0 = \rho$ and that is contained in $\mathcal{O}(\rho)$. That means that there exists a path g_t in G such that $g_0 = id$ and

$$\rho_t(\cdot) = g_t \rho(\cdot) g_t^{-1}.$$

As before, we are just interested in the first order approximation of the path. Then we can say that $g_t = e^{tX} + o(t)$ for an $X \in \mathfrak{g}$ and $\rho_t(\gamma) = e^{tc(\gamma)}\rho_0(\gamma) + o(t)$. Substituting these expressions and computing the derivative in $t = 0$ we get

$$c(\gamma)\rho(\gamma) = X\rho(\gamma) - \rho(\gamma)X,$$

that means

$$c(\gamma) = X - \rho(\gamma)X\rho(\gamma)^{-1} = X - \operatorname{Ad}_{\rho(\gamma)} X.$$

Cohomological Interpretation

As maybe the reader has already noticed, there is a direct link between the previous two sections. In fact a representation ρ equips $\mathfrak{g}$ with the structure of a Γ-module by the action $\Gamma \to \operatorname{Aut}(\mathfrak{g})$ given by

$$\Gamma \xrightarrow{\ \rho\ } G \xrightarrow{\ \operatorname{Ad}\ } \operatorname{Aut}(\mathfrak{g})$$

$$\gamma \longmapsto \rho(\gamma) \longmapsto \operatorname{Ad}_{\rho(\gamma)}$$

The resulting Γ-module is denoted by $\mathfrak{g}_\rho$.
Therefore we have this wonderful identifications

$$\begin{aligned} Z^1(\Gamma, \mathfrak{g}_\rho) &= \{c \in C^1 \mid c(\gamma_1\gamma_2) = \operatorname{Ad}_{\rho(\gamma_1)} c(\gamma_2) + c(\gamma_1)\} = T_\rho \operatorname{Hom}(\Gamma, G) \\ B^1(\Gamma, \mathfrak{g}_\rho) &= \{c \in C^1 \mid c(\gamma) = \operatorname{Ad}_{\rho(\gamma)} X - X, \text{ for a } X \in \mathfrak{g}\} = T_\rho \mathcal{O}(\rho). \end{aligned}$$

and, following what we said before it holds that

$$T_\rho \operatorname{Hom}(\Gamma, G)/T_\rho \mathcal{O}(\rho) = H^1(\Gamma, \mathfrak{g}_\rho).$$

We stress that for the moment we have not shown that it is the tangent space at a point of the character variety, but it will follow from what we will say in the next section.

1.5 Smooth Points

Up to now we have built a tangent space for any smooth point of the character variety. The aim of this section is to show that the work we did is not wasted since there are a lot of smooth points in the character variety. In particular smooth points arise when the action is proper and locally free (that means that in a neighbourhood of that point we do not need the GIT quotient). We recall that an action is free when there are no fixed points and it is proper if the function $(g, x) \mapsto (g \cdot x, x)$ is proper.

Definition. The action of a topological group on a set X is *locally free* at x if the stabiliser of x is discrete.

Lemma 1.3. *Let $Z(\rho)$ be the stabiliser of $\rho \in \mathrm{Hom}(\Gamma, G)$. The $Inn(G)$-action is free on the set of representations such that $Z(\rho) = Z(G)$ and it is locally free on the set of representations such that $\dim Z(\rho) = \dim Z(G)$.*

Proof. The action induces, for any representation ρ, a surjective linear map $\mathfrak{Inn}(G) \to T_\rho O_\rho$ given by

$$\xi \mapsto \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot \rho.$$

The $Inn(G)$-action is locally free if and only if the map is also injective. Since the map is always surjective, it suffices to ask that both spaces $\mathfrak{Inn}(G)$ and $T_\rho O_\rho$ have the same dimension. The first one has dimension $\dim G - \dim Z(G)$. We recall that we have the identification

$$T_\rho O_\rho \cong H^1(\Gamma, \mathfrak{g}_\rho) = \{c \in C^1 \mid c(\gamma) = \mathrm{Ad}_{\rho(\gamma)} X - X, \text{ for a } X \in \mathfrak{g}\}.$$

In particular this definition states that we can see this space as the quotient $\mathfrak{g}/\mathfrak{z}(\rho)$, where $\mathfrak{z}(\rho) = \mathrm{Lie}(Z(\rho))$ and then its dimension is $\dim G - \dim Z(\rho)$. Hence the dimensions coincide if and only if $\dim Z(G) = \dim Z(\rho)$. □

We will soon see that irreducible representations will play a crucial role in this story. Let so us recall the definition and some facts about them.

Definition. Let G be $\mathrm{SL}(n, \mathbb{C})$. A representation $\rho \in \mathrm{Hom}(\Gamma, G)$ is *irreducible* if it does not fix any proper subspace of $\mathbb{C}^n$. We will denote by $\mathrm{Hom}^{irr}(\Gamma, G)$ the set of irreducible representations.

Proposition 1.4. *If S is a surface of genus grater than 1, the set $\mathrm{Hom}^{irr}(\Gamma, G)$ is a Zarisky-open in $\mathrm{Hom}(\Gamma, G)$, and it is non empty if Γ is a free group or a surface group.*

This result is important for us because it states that irreducible representations form a dense subspace of all the representations. Moreover in the special case $G = \mathrm{SL}(2, \mathbb{C})$ we have the following result that characterise irreducible representations in terms of trace.

Proposition 1.5. *A representation ρ is irreducible if and only if there exists an element $\gamma \in [\Gamma, \Gamma] \subseteq \Gamma$ such that $\mathrm{Tr}(\rho(\gamma)) \neq 2$.*

Proof. See [6], Lemma 2.7. □

We conclude the section with this crucial result, that make a link between irreducible representation and the smooth points of the character variety.

Theorem 1.6. *The $Inn(G)$-action on $\mathrm{Hom}^{irr}(\Gamma, G)$ is locally free and proper.*

Then, if ρ is irreducible, the tangent space

$$T_{[\rho]}\chi(\Gamma, G) = T_\rho \mathrm{Hom}(\Gamma, G)/T_\rho O(\rho) = H^1(\Gamma, \mathfrak{g}_\rho)$$

is well defined.

1.6 The $\mathrm{MCG}^*(S)$-action

In this last part the bridge between mapping class groups and character varieties is built.
First of all we recall that for a generic group Γ, we have that $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/Inn(\Gamma)$ is the group of exterior automorphisms. Clearly if Γ is abelian we have that $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)$ because all inner automorphisms are trivial. Let us now consider Γ as a surface group. In particular it is realised as the fundamental group of a punctured surface $S = S_{g,b}$ and hence $\Gamma \cong \pi_1(S, p)$ for a point $p \in S$. Let now $f \in \mathrm{Homeo}(S)$ be a homeomorphism of S. From the algebraic topology, we know that f induces a isomorphism between the fundamental groups

$$\begin{aligned} f_*\colon \pi_1(S,p) &\to \pi_1(S, f(p)) \\ [\gamma] &\longmapsto [f \circ \gamma] \end{aligned}$$

Moreover, since S is path connected, we can always find a path β between $f(p)$ and p and consider the induced inner automorphism

$$\begin{aligned} \Theta_\beta\colon \pi_1(S,f(p)) &\to \pi_1(S, p) \\ [\gamma] &\longmapsto [\beta \star \gamma \star \bar{\beta}] \end{aligned}$$

where we denoted with $\star$ the product of paths and with $\bar{\beta}$ the inverse path. In conclusion, given a homeomorphism f of S and a path β between $f(p)$ and p we are able to build an automorphism of the fundamental group $\pi_1(S, p)$:

$$\begin{aligned} f_*^\beta := \Theta_\beta \circ f_*\colon \pi_1(S, p) &\to \pi_1(S, p) \\ [\gamma] &\longmapsto [\beta \star (f \circ \gamma) \star \bar{\beta}], \end{aligned}$$

and then we got a map $\mathrm{Homeo}(S) \to \mathrm{Aut}(\Gamma)$, $f \mapsto f_*^\beta$.
Up to now we have a sort of idea for building our bridge. We have anyway to face some problems:

- *A priori*, the map we defined depends on the choice of β and then it is not a group morphism.
- We would like to define the same morphism but with domine $\mathrm{MCG}^*(S)$ and not the whole group of homeomorphisms .
- We are interested in $\mathrm{Out}(\Gamma)$ and not the whole group of automorphisms.

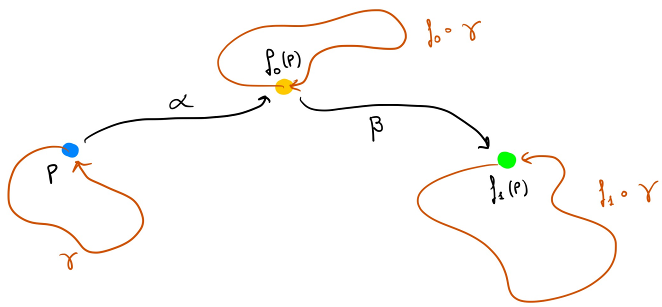
As (incredibly) often happens in mathematics the problems all together provide a solution: it turns out that changing the path β in an another one β' will give an another automorphism of Γ that is conjugated to the first one, in symbols: $f_*^\beta = \Theta_{\beta\star\bar{\beta}'} \circ f_*^{\beta'}$. Therefore $[f_*^\beta] = [f_*^{\beta'}]$ and we have a well defined morphism $\mathrm{Homeo}(S) \to \mathrm{Out}(\Gamma)$ that does not depend on the choice of β. Finally we would like to show that if f is isotopic to g, then they will induce the same element in $\mathrm{Out}(\Gamma)$, that is, build the following commutative diagram.

$$\begin{tikzcd} \mathrm{Homeo}(S) \arrow[d, "f \mapsto [f]_{\text{isotopy}}"'] \arrow[dr, "f \mapsto [f_*^\beta]"] & \\ \mathrm{MCG}^*(S) \arrow[r, dotted] & \mathrm{Out}(\Gamma) \end{tikzcd}$$

Let us give a name to the morphism $\Phi\colon \mathrm{Homeo}(S) \to \mathrm{Out}(\Gamma)$.

Proposition 1.7. *If $f_0 \sim f_1 \in \mathrm{Homeo}(S)$, then $\Phi(f) = \Phi(g) \in \mathrm{Out}(\Gamma)$.*

Proof. We recall that $\Gamma \cong \pi_1(S, p)$. If $f_0 \sim f_1$, then there exists $H\colon t \mapsto f_t$ isotopy between them. Then for any $[\gamma] \in \Gamma$, we have that $f_0 \circ \gamma$ is homotopic to $f_1 \circ \gamma$. Moreover we have that $f_0 \circ \gamma$ is homotopic to $\beta \star (f_1 \circ \gamma) \star \bar{\beta}$, where β is a path between $f_0 \circ \gamma(0) = f_0(p)$ and $f_1 \circ \gamma(0) = f_1(p)$.
We want to show that $[(f_0)_*^{\alpha}] = [(f_1)_*^{\alpha\star\beta}]$ where α is a path between p and $f_0(p)$, as shown in the following figure.



In fact:

$$\begin{aligned}(f_1)_*^{\alpha\star\beta}[\gamma] &= [\alpha \star \beta \star (f_1 \circ \gamma) \star \overline{\alpha \star \beta}] \\ &= [\alpha \star \beta \star (f_1 \circ \gamma) \star \bar{\beta} \star \bar{\alpha}] \\ &= [\alpha \star (f_0 \circ \gamma) \star \bar{\alpha}] \\ &= (f_0)_*^{\alpha}[\gamma].\end{aligned}$$

□

Moreover we have these powerful and crucial results.

Theorem 1.8 (Dehn-Nielsen-Baer). *If $S = S_{g,0}$ is a closed surface, the application induced on the quotient*

$$\tilde{\Phi}\colon \mathrm{MCG}^*(S) \to \mathrm{Out}(\Gamma)$$

is an isomorphism of groups.

Proof. We have already seen it is well defined. For a proof of the bijectivity, see [3], chapter 8. □

Theorem 1.9. *If $S = S_{g,b}$ is a punctured surface, the application induced on the quotient*

$$\tilde{\Phi}\colon \mathrm{MCG}^*(S) \to \mathrm{Out}^\star(\Gamma)$$

is an isomorphism of groups. Where $\mathrm{Out}^\star(\Gamma)$ is the subgroup of $\mathrm{Out}(\Gamma)$ that preserves the set of conjugacy classes of the generators $c_i \in \Gamma$.

Proof. See [3], theorem 8.8. □

These theorems are a turning point for us: in the following sections we want to study, in the special case $S = S^2_4$, the action of the extended mapping class group on the character variety. The two principal ingredients to do so are then the Dehn-Nielsen-Baer theorem (DNB) and the action $\mathrm{Aut}(\Gamma) \curvearrowright \mathrm{Hom}(\Gamma, G)$ that we recall is given, for $\varphi \in \mathrm{Aut}(\Gamma)$, by.

$$\varphi(\rho)(\gamma) := \rho(\varphi^{-1}(\gamma)).$$

As before, we have some issues to solve because the DNB theorem considers only outer automorphisms. Moreover we are interested in the action on the character variety and not on the whole representation variety.
As a miracle, as before, all this issue perfectly fits in a solution. In fact by a simple calculation we get that if $\varphi = C_\beta \in Inn(\Gamma)$, $\rho \in \mathrm{Hom}(\Gamma, G)$ and $\gamma \in \Gamma$, then:

$$\varphi(\rho)(\gamma) = \rho(\varphi^{-1}(\gamma)) = \rho(\bar{\beta} \star \gamma \star \beta) = \rho(\beta)^{-1} \cdot \rho(\gamma) \cdot \rho(\beta) = C_{\rho(\beta)^{-1}} \circ \rho(\gamma).$$

And this implies that $Inn(\Gamma)$ acts trivially on $\mathrm{Hom}(\Gamma, G)/Inn(G)$ and we can consider the following well defined commutative diagram.

$$\begin{tikzcd}
\mathrm{Aut}(\Gamma) \arrow[d, "proj"'] & \curvearrowright & \mathrm{Hom}(\Gamma, G) \arrow[d, "proj"] \\
\mathrm{Out}(\Gamma) & \curvearrowright & \mathrm{Hom}(\Gamma, G)/Inn(G)
\end{tikzcd}$$

A more subtle argument (that involves some technical details we have not presented) shows that actually this action descends also to the character variety $\chi(\Gamma, G)$ and then we have a well defined action

$$\mathrm{MCG}^*(S) \cong \mathrm{Out}(\Gamma) \curvearrowright \chi(\Gamma, G)$$

given by

$$\begin{array}{ccc}
\mathrm{MCG}^*(S) & \longrightarrow & \mathrm{Aut}(\chi(\Gamma, G)) \\
{[f]}_{\text{isotopy}} & \longmapsto & \{\chi(\rho) \mapsto \chi(\rho \circ f^{-1})\}.
\end{array}$$

2 The Character Variety $\chi(\mathbb{S}^2_4)$ and the MCG-Action

Our main goal is to describe the action of the extended mapping class group of the 4-punctured sphere on its representation variety onto SL(2, $\mathbb{C}$). To do so we will briefly describe the fundamental group of $\mathbb{S}^2_4$ and recall some properties of the trace that will be important to study the representation variety.

Fundamental group

We can visualise the 4-punctured sphere in this way

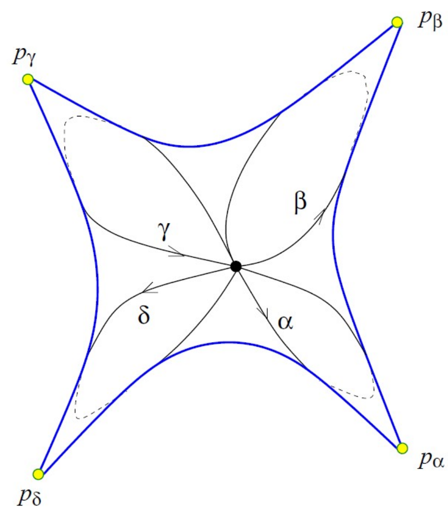


Figure 1: The four punctured sphere, [2], page 3.

where α, β, γ and δ are the generators of the fundamental group (what we called c_i before) and $p_\alpha, p_\beta, p_\gamma$ and p_δ are the punctures.

Remark. Sometimes will be more practical for us to see $\mathbb{S}^2$ as $\mathbb{CP}^1$ and $(p_\alpha, p_\beta, p_\gamma, p_\delta) = (0, 1, t, \infty)$.

By what we have said in the previous chapter (see Example 2) we have that:

$$\pi_1(\mathbb{S}^2_4) = \pi_1(S_{0,4}) = \langle \alpha, \beta, \gamma, \delta \ \mid \ \alpha\beta\gamma\delta = 1\rangle$$

In particular we have that given three out of four of the generators, the last one is given, for example, by the formula $\delta = (\alpha\beta\gamma)^{-1}$. This implies that the only relation we have suffices to reconstruct the fourth generator, given the others. In particular we can delete the relation and a generator in order to get a free group:

$$\pi_1(\mathbb{S}^2_4) \cong \mathbb{F}_3 = \langle \ \alpha, \beta, \gamma \ \rangle.$$

Traces properties

We collect here some results that may help the reader to follow the sequel. For any proof and a more detailed exposition of the subject we refer to [5].

It is a standard fact that the trace is a commutative operator, but not a multiplicative morphism. In particular these three formulas holds:

- $\mathrm{Tr}(uv) = \mathrm{Tr}(u)\,\mathrm{Tr}(v) - \mathrm{Tr}(uv^{-1})$,
- $\mathrm{Tr}(uvw) = \mathrm{Tr}(u)\,\mathrm{Tr}(vw) + \mathrm{Tr}(v)\,\mathrm{Tr}(uw) + \mathrm{Tr}(w)\,\mathrm{Tr}(uv) - \mathrm{Tr}(u)\,\mathrm{Tr}(v)\,\mathrm{Tr}(w) - \mathrm{Tr}(uwv)$,
- $\mathrm{Tr}(u^m) = \mathrm{Tr}(u^{m-1})\,\mathrm{Tr}(u) - \mathrm{Tr}(u^{m-2})$,
- $\mathrm{Tr}(u) = \mathrm{Tr}(u^{-1})$.

The importance of the trace function raised up when we tried to make the quotient $\mathrm{Hom}(\Gamma, G)/Inn(G)$ look better. In particular we decided to identify two orbits in the quotient if they have the same image in $\mathbb{C}$ under the trace and it turned out that this procedure gives a complex algebraic variety (maybe with singularities). Therefore we should explore more in details the relationship between the trace function and the conjugacy classes.
Before starting, just a little notation interlude: when we will talk about the trace of a word u that belongs to the free group on n-generators $\mathbb{F}_n$, we mean its trace under an arbitrary representation $\rho\colon \mathbb{F}_n \to \mathrm{SL}(2,\mathbb{C})$.

Example 5. In particular we can start by saying that for $\mathbb{F}_2 = \langle a, b\rangle$ there exists some words which are not conjugated, but with the same trace. For instance:

- $\mathrm{Tr}(a^2b^{-1}ab) = \mathrm{Tr}(a^2bab^{-1})$
- $\mathrm{Tr}(ab^2a^2b) = \mathrm{Tr}(aba^2b^2)$

Then (as the different topology of the two quotients has already shown) it makes sense to investigate the problem.

Lemma 2.1. *If $u, u' \in \mathbb{F}_2$ are two cyclically reduced words such that $\mathrm{Tr}(u) = \mathrm{Tr}(u')$, then any generator occurs exactly as many times in u and in u' (up to a choice of ± 1 at the exponent).*

Theorem 2.2. *If a is a generator and $\mathrm{Tr}(u) = \mathrm{Tr}(a^m)$ for $m > 0$, then u is conjugated to a^m or a^{-m}.*

The following last result is quite important. In fact we may ask ourselves how many the "second" quotient by the trace function is changing the original one. In particular we do not want a huge amount of orbits to be identified.

Theorem 2.3. *Every character class in a free group is the union of a finite number of conjugacy classes.*

2.1 Representation Variety and Character Variety

Let $\rho \in \mathrm{Hom}\left(\pi_1(\mathbb{S}^2_4), \mathrm{SL}(2,\mathbb{C})\right) := \mathrm{Rep}(\mathbb{S}^2_4)$ a representation of the fundamental group of the 4-punctured sphere onto $\mathrm{SL}(2,\mathbb{C})$. We may ask how many data we need to completely reconstruct ρ in terms of matrices and in terms of traces.
By definition, ρ is completely determined by its values on the generators. However, we can do better: since ρ is a group morphism (then it preserves the relations), if we consider $\pi_1(\mathbb{S}^2_4) \cong \mathbb{F}_3$, we can

reconstruct ρ starting by $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\gamma)$ getting that $\rho(\delta) = \rho((\alpha\beta\gamma)^{-1})$.
In terms of traces we can set

$$a = \mathrm{Tr}(\rho(\alpha)) \qquad b = \mathrm{Tr}(\rho(\beta)) \qquad c = \mathrm{Tr}(\rho(\gamma)) \qquad d = \mathrm{Tr}(\rho(\delta)),$$

but, as we have seen, we can not uniquely derive one by the others. In fact:

$$d = \mathrm{Tr}(\rho(\alpha\beta\gamma)^{-1}) = \mathrm{Tr}(\rho(\alpha\beta\gamma)) = a\,\mathrm{Tr}(\rho(\beta\gamma)) + b\,\mathrm{Tr}(\rho(\alpha\gamma)) + c\,\mathrm{Tr}(\rho(\alpha\beta)) - abc - \mathrm{Tr}(\rho(\alpha\gamma\beta)). \tag{1}$$

Therefore we need at least 4 coordinates a, b, c, d in $\mathbb{C}$ to reconstruct ρ. Are they sufficient? No. In fact we can choose ρ' such that it takes the same value as ρ on α and β and different values on γ and δ such that the traces are the same, i.e. $(a, b, c, d) = (a', b', c', d')$.

Example 6. Let us consider the representations ρ and ρ' defined as follows on the generators.

$$\begin{array}{ccccc} & \alpha & \beta & \gamma & \delta \\ \\ \rho & Id & Id & Id & Id \\ \rho' & Id & Id & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{array}$$

In this case $(a, b, c, d) = (a', b', c', d') = (2, 2, 2, 2)$.

Anyway, as formula (1) suggests, we can introduce three more coordinates

$$x = \mathrm{Tr}(\rho(\alpha\beta)) \qquad y = \mathrm{Tr}(\rho(\beta\gamma)) \qquad z = \mathrm{Tr}(\rho(\gamma\delta)).$$

Then we have built the following coordinate function

$$\begin{array}{cccc} \chi\colon & \mathrm{Rep}(\mathbb{S}^2_4) & \to & \mathbb{C} \\ & \rho & \mapsto & (a, b, c, d, x, y, z) \end{array}$$

that is clearly invariant for conjugation and its components satisfy the equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D, \tag{2}$$

where

$$A = ab + cd, \qquad B = bc + ad, \qquad C = ac + bd, \qquad D = 4 - a^2 - b^2 - c^2 - d^2 - abcd.$$

Actually these seven coordinates are sufficient to reconstruct a representation ρ up to its character class. That means that:

Theorem 2.4. *The quotient* $\mathrm{Rep}(\mathbb{S}^2_4)//Inn(\mathrm{SL}(2, \mathbb{C})) := \chi(\mathbb{S}^2_4)$ *is isomorphic to the six-dimensional quartic hypersurface of* $\mathbb{C}^7$ *given by the equation 2.*

Example 7. We see that if we consider ρ and ρ' as in the previous example, we get also that $(x, y, z) = (x', y', z') = (2, 2, 2)$. Anyway we easily see that the two representations are not conjugated, this shows that with these seven parameters we are able to go back to the character class of ρ and not to its conjugacy class.

Remark. The theorem is coherent with the fact that the algebra of polynomial functions on $\mathrm{Rep}(\mathbb{S}^2_4)$ which are invariant under conjugation is generated by the trace functions, i.e. the components of the function χ.

It is not by chance that we gave to the coordinates two different type of names. In fact, as the formulation of the equation 2 suggests, we will focus our interest in studying that manifold when A, B, C and D are fixed. This give rise to a cubic surface, that will be denoted by $S_{(A,B,C,D)}$, in $\mathbb{C}^3$ of coordinates x, y, z.

Remark. The map

$$\begin{array}{ccc} \mathbb{C}^4 & \to & \mathbb{C}^4 \\ (a,b,c,d) & \mapsto & (A,B,C,D) \end{array}.$$

is a non Galois ramified cover of degree 24. For more properties of this map and an accurate study of the fibers we refer to [2], Appendix B.

Definition. We will denote by Fam the set of all surfaces of type $S_{(A,B,C,D)}$ defined by the equation 2 for different values of parameters.

Therefore, to sum up, we have the varieties

- $\chi(\mathbb{S}^2_4) \cong \{(a,b,c,d,x,y,z) \in \mathbb{C}^7 \mid x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D\} \subseteq \mathbb{C}^7$,
- $S_{(A,B,C,D)} := \{(x,y,z) \in \mathbb{C}^3 \mid x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D\} \subseteq \chi(\mathbb{S}^2_4)$.

and the following maps

$$\begin{array}{ccccccccc} \mathrm{Rep}(\mathbb{S}^2_4) & \xrightarrow{\chi} & \chi(\mathbb{S}^2_4) \subseteq \mathbb{C}^7 & \xrightarrow{proj} & \mathbb{C}^4 & \xrightarrow{deg\ 24} & \mathbb{C}^4 & \longrightarrow & Fam \\ \rho & \longmapsto & (a,b,c,d,x,y,z) & \longmapsto & (a,b,c,d) & \longmapsto & (A,B,C,D) & \longmapsto & S_{(A,B,C,D)}. \end{array}$$

Before the description of the mapping class group action, we would like to illustrate how the different surfaces $S_{(A,B,C,D)}$ are made. To do so let us give a look at their real part in the following picture.

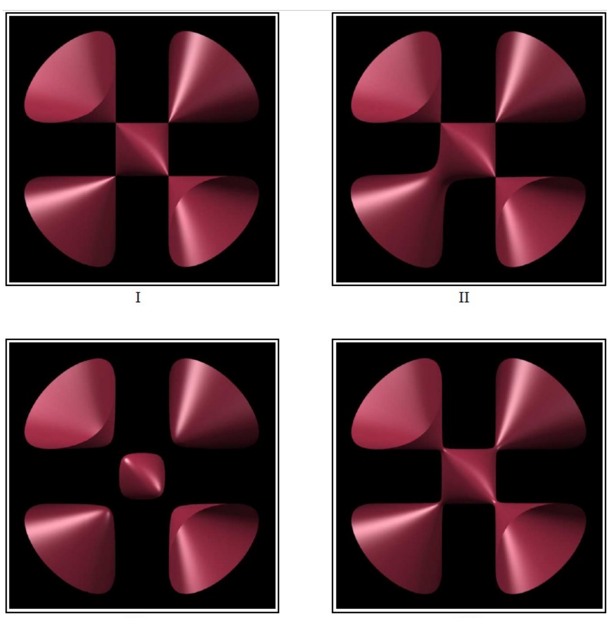


Figure 2: Examples, [2], page 8.

The image (I) is the so called Cayley cubic and it corresponds to $S_{(0,0,0,4)}$. It is the only element in Fam in which 4 singularities occur, while the others reach at most 3 singularities, as in (II), $S_{(-0.2,-0.2,-0.2,4.39)}$. To complete the description, we have that (III) corresponds to $S_{(0,0,0,3)}$ and (IV) corresponds to $S_{(0,0,0,4.1)}$.

The Cayley cubic is then a simple and quite important member of Fam. Before going on we can spend some words about it and study the dynamics of the mapping class group on it.

Theorem 2.5 (Cayley)**.** *If $S \in Fam$ has four singular points, then it coincides with the Cayley cubic.*

The Cayley cubic can be realised as the quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution $\eta(u, v) = (u^{-1}, v^{-1})$. The map

$$\pi_C(x, y) = \left(-u - \frac{1}{u}, -v - \frac{1}{v}, -uv - \frac{1}{uv} \right)$$

gives an explicit isomorphism between the Cayley cubic and $(\mathbb{C}^* \times \mathbb{C}^*)/\eta$. The four fixed points of this involution, that are $(1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$, respectively corresponds to the four singular points of the Cayley cubic.

The group $\mathrm{GL}(2, \mathbb{Z})$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ by monomial transformation and this action commutes with η, permuting its fixed points. As a consequence, $\mathrm{PGL}(2, \mathbb{Z})$ acts on the Cayley cubic. Precisely, the generators acts this way:

- $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : (x, y, z) \to (z, y, -x - yz)$
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : (x, y, z) \to (x, -z - xy, y)$
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : (x, y, z) \to (x, y, -z - xy)$.

where in blue are stressed the fixed element of the action.

We will see in Lemma 2.7 that this action extends in a natural way to any $S \in Fam$ and we will explain in that more general case where these formulae come from.

2.2 The Mapping Class Group and its Action

As we have seen in the first chapter, the DNB theorem states that there is an isomorphism between $\mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4))$ and the extended mapping class group. In our case we have:

$$\mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4)) \cong \mathrm{Out}^\star(\mathbb{F}_3) \cong \mathrm{MCG}^*(\mathbb{S}^2_4),$$

and the induced action

$$\begin{array}{ccc} \mathrm{MCG}^*(\mathbb{S}^2_4) & \longrightarrow & \mathrm{Aut}(\chi(\mathbb{S}^2_4)) \\ [f]_{\text{isotopy}} & \longmapsto & \{\chi(\rho) \mapsto \chi(\rho \circ f^{-1})\}. \end{array}$$

Our goal for this section is to better describe the mapping class group of the 4-punctured sphere and to give some explicit formulae to its action on the character variety and describe the subgroup which stabilises each surface $S_{(A,B,C,D)} \subseteq \chi(\mathbb{S}^2_4)$.

Let us consider the two-folded ramified covering $\pi_T\colon T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{S}^2$ with Galois involution

$$\sigma\colon (x, y) \mapsto (-x, -y)$$

sending the set ramification points $Q = \{(\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (0, 0)\}$ to the four punctures of the sphere which we will call respectively $p_\alpha, p_\beta, p_\gamma$ and p_δ. In particular σ acts on the square $[0, 1] \times [0, 1]$ as a symmetry centred in $(\frac{1}{2}, \frac{1}{2})$, that is then a fixed point. Any point in the interior of the square has an orbit composed of two distinct points in the torus under σ, while all the vertices (that are actually the same point on the torus) are sent one in the other and then $(0, 0) \in T^2$ is fixed. Moreover $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are respectively sent in the other point of the square that belongs to their equivalent class under the $\mathbb{Z}^2$-action, and then they result to be fixed by σ on the torus. In particular this ramified covering establish an homeomorphism between the torus deprived of the ramification points and the 4-punctured sphere.
The extended mapping class group of the torus is isomorphic to $\mathrm{GL}(2, \mathbb{Z})$ and it acts by linear homeomorphisms fixing $(0, 0)$ (this implies that $\mathrm{GL}(2, \mathbb{Z})$ is also the mapping class group of the 1-punctured torus $T^2 \setminus \{(0, 0)\}$).

Remark. We stress the evident fact that the non-extended mapping class group of the torus is then $\mathrm{SL}(2, \mathbb{Z})$, composed of matrices of determinant equal to one.

This action commutes with the involution: in fact if $p = (x, y) \in T^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$, we have that

$$\sigma(Ap) = \sigma(ax + by, cx + dy) = (-ax - by, -cx - dy) = A\sigma(p).$$

Since moreover σ generates the centre of $\mathrm{GL}(2, \mathbb{Z})$ and the set Q is invariant under the action of $GL(2, \mathbb{Z})$, we get a morphism

$$\mathrm{MCG}^*(T^2)/\sigma \cong \mathrm{GL}(2, \mathbb{Z})/Z(\mathrm{GL}(2, \mathbb{Z})) \cong \mathrm{PGL}(2, \mathbb{Z}) \to \mathrm{MCG}^*((T^2 \setminus Q)/\sigma) \cong \mathrm{MCG}^*(\mathbb{S}^2_4).$$

This morphism is injective but not surjective: in fact it turns out that its image is the stabiliser of p_δ, that we recall is the image of $(0, 0)$ under the covering map we fixed at the beginning.

Remark. The fact that the action of $\mathrm{GL}(2, \mathbb{Z})$ commutes with σ is necessary to let this action pass to the quotient $(T^2 \setminus Q)/\sigma \cong \mathbb{S}^2_4$.

We then have $\mathrm{PGL}(2, \mathbb{Z}) \subseteq \mathrm{MCG}^*(\mathbb{S}^2_4)$, that is the first ingredient to build our mapping class group. To complete it let us have a look at H, the 2-torsion subgroup of $(T^2, +)$ given by the the ramification points of π_T: $(\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ and $(0, 0)$. It acts on the group $(T^2, +)$ via translation and commmutes via the involution σ, then it pass to the quotient. As a consequence of this, it turns out that

Proposition 2.6. *The morphism of groups*

$$\mathrm{PGL}(2, \mathbb{Z}) \ltimes H \to \mathrm{MCG}^*(\mathbb{S}^2_4)$$

is actually an isomorphism.

Proof. For a detailed proof see [1], section 4.4. □

We gave a more explicit description of the mapping class group and we can then more easily study its action on the character variety $\chi(\mathbb{S}^2_4)$.

- As we have briefly seen before, the image of $\mathrm{PGL}(2,\mathbb{Z})$ inside the mapping class group coincide with the stabiliser of $p_\delta = \pi_T(0,0)$. Moreover it freely permutes the other three ramification points.
- The image of H permutes all the four punctures by products of disjoint transpositions and acts trivially on $\chi(\mathbb{S}^2_4)$. In fact we recall that the character variety is given by the equation 2 and H acts trivially on (A,B,C,D,x,y,z), because it actually only permutes the parameters (a,b,c,d).

Therefore, the action $\mathrm{MCG}^*(\mathbb{S}^2_4) \curvearrowright \chi(\mathbb{S}^2_4)$ is reduced to the action of $\mathrm{PGL}(2,\mathbb{Z})$ on it.

Lemma 2.7. *The subgroup of* $\mathrm{Aut}(\chi(\mathbb{S}^2_4))$ *given by the action of* $\mathrm{PGL}(2,\mathbb{Z}) \subseteq \mathrm{MCG}^*(\mathbb{S}^2_4)$ *is generated by*

- $B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{cases} (a,b,c,d) \to (b,a,c,d) \\ (x,y,z) \to (x, -z-xy+ac+bd, y) \end{cases}$
- $B_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : \begin{cases} (a,b,c,d) \to (a,c,b,d) \\ (x,y,z) \to (z, y, -x-yz+ab+cd) \end{cases}$
- $T_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \begin{cases} (a,b,c,d) \to (c,b,a,d) \\ (x,y,z) \to (y,x,z) \end{cases}$

where in blue are emphasised the fixed elements.

Proof. See [2], Lemma 2.2. ◻

Remark. This result is very important and we have a lot to say about it.

- First of all, let us notice that the action of $\mathrm{PGL}(2,\mathbb{Z})$ on $\chi(\mathbb{S}^2_4)$ is polynomial.
- Then we notice that the two polynomials that appears in the formula have the same structure, then we may wonder where they come from. We analyse B_1, but for B_2 is the same reasoning. We recall that the relation of the coordinates is described by a second degree equation in x, y and z. Since we are working in $\mathbb{C}$, for any choice of a,b,c,d, there are two solution for any x,y,z, counted with multiplicity. In particular, if we suppose to have fixed x and y, it holds that if z_0 and z_1 are the solutions, then their sum is given by the sum of the coefficients of the linear part of the polynomial. In formulae

  $$z_0 + z_1 = -xy + C = -xy + ac + bd \quad \text{that means} \quad z_0 = -z_1 - xy + C = -z_1 - xy + ac + bd$$

  that is exactly the polynomial that appears in the formula of B_1, that then fixes x, sends z to the other solution of the polynomial and finally exchanges the new z and y.
- As promised, this lemma extends what we have said for the Cayley cubic, that we recall it is $S_{(0,0,0,4)}$.

2.3 The Subgroup Γ_2^*

The reader may wonder why we have introduced the surfaces $S_{(A,B,C,D)}$. For instance, we will see that the action of the extended mapping class group on the character variety is (almost) the same action of $\mathrm{Aut}[S_{(A,B,C,D)}]$ on $S_{(A,B,C,D)}$. To do so we are going to analyse a particular subgroup of $\mathrm{MCG}^*(\mathbb{S}_4^2)$ and its dynamics.

First of all, we notice that, in the end of the previous section, we studied the formulae that appear in Lemma 2.7. In particular, we should focus on these three involutions

- $s_x = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} : (x, y, z) \to (-x - yz + A, y, z)$
- $s_y = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : (x, y, z) \to: (x, -y - xz + B, z)$
- $s_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : (x, y, z) \to (x, y, z - xy + C).$

Definition. We will denote by Γ_2^* the subgroup of $\mathrm{PGL}(2, \mathbb{Z}) \hookrightarrow \mathrm{MCG}^*(\mathbb{S}_4^2)$ given by

$$\Gamma_2^* := \langle s_x, s_y, s_z \rangle$$

It is pretty clear that s_x, s_y and s_z reverse the orientation.

Definition. We will denote by Γ_2 the subgroup of $\mathrm{PGL}(2, \mathbb{Z}) \hookrightarrow \mathrm{MCG}^*(\mathbb{S}_4^2)$ given by

$$\Gamma_2 := \langle g_x := s_z s_y, \quad g_y := s_x s_z, \quad g_z = s_y s_x \rangle$$

It results that $\Gamma_2 \subseteq \Gamma_2^*$ is exactly the subgroup of orientation preserving functions.
The name Γ_2^* sounds bizarre. We will then explain where it comes from. The action of $M \in \mathrm{GL}(2, \mathbb{Z})$ on the set $H \subseteq T^2$ of the fixed point of σ only depends on the equivalence class of M modulo 2. To see that, is way harder to use words than showing an example:

$$\begin{tikzcd}[column sep=small]
 & (0,0) \arrow[r, Rightarrow, "{a=c=2}"] & (1,1) \\
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \left(\frac{1}{2}, 0\right) \mapsto \left(\frac{1}{2}a, \frac{1}{2}c\right) \equiv mod(2) \arrow[ur, "PP"] \arrow[r, "PU"] \arrow[dr, "UP"] \arrow[ddr, "UU"'] & (0, \frac{1}{2}) \arrow[r, Rightarrow, "{a=2,c=1}"] & (1, \frac{1}{2}) \\
 & (\frac{1}{2}, 0) \arrow[r, Rightarrow, "{a=1,c=2}"] & (\frac{1}{2}, 1) \\
 & (\frac{1}{2}, \frac{1}{2}) \arrow[r, Rightarrow, "{a=c=3}"] & (\frac{3}{2}, \frac{3}{2})
\end{tikzcd}$$

We then get an exact sequence

$$I \to \Gamma_2^* \to \mathrm{PGL}(2, \mathbb{Z}) \ltimes H \to Sym_4 \to 1$$

where Γ_2^* is the subgroup of those matrices $M \equiv I \bmod(2)$. It turns out to be the same Γ_2^* seen before. Therefore the generators of the group Γ_2^* are defined by exchanging the two roots of the polynomial defining $S_{(A,B,C,D)}$ and all the elements of this group fix the punctures and preserve a, b, c and d. We would like to create a link between the mapping class group of the 4-punctured sphere and the group of polynomial automorphisms of a surface $S_{(A,B,C,D)}$. The assist for that comes exactly from the fact that Γ_2^* preserves a, b, c and d and then preserves any $S_{(A,B,C,D)}$.

Proposition 2.8. *Let $\mathrm{MCG}_0^*(\mathbb{S}_4^2)$ the subgroup which stabilises the four punctures of $\mathbb{S}_4^2$. This group coincides with the stabiliser of the projection*

$$\begin{array}{rccc} \pi: & \chi(\mathbb{S}_4^2) & \to & \mathbb{C}^4 \\ & (a,b,c,d,x,y,z) & \mapsto & (a,b,c,d) \end{array}$$

and its image in $\mathrm{Aut}(\chi(\mathbb{S}_4^2))$ coincides with the image of Γ_2^* and is therefore generated by the tree involutions s_x, s_y and s_z.

Remark. If we moreover want the functions to preserve the orientation we can consider the same proposition with $\mathrm{MCG}_0(\mathbb{S}_4^2)$, Γ_2 and g_x, g_y and g_z.

This result is very important: in fact it states that the action of the subgroup of the extended mapping class group that fixes the surfaces $S_{(A,B,C,D)}$ can be reduced to the action of $\Gamma_2^* = \langle s_x, s_y, s_z\rangle$ and we can then study this action for which we already have some formulae.

Theorem 2.9. *Let $S = S_{(A,B,C,D)} \in Fam$. Then*

- *there is no trivial relation between s_x, s_y and s_z and therefore*

$$\Gamma_2^* \cong (\mathbb{Z}/2\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z}),$$

- *the index of Γ_2^* in $\mathrm{Aut}[S]$ is bounded by 24.*

Moreover, for a generic choice of the parameters (A, B, C, D), Γ_2^* is equal to $\mathrm{Aut}[S]$.

Proof. See [2], Theorem 3.1. □

In other words, up to the finite index subgroups, describing the dynamics of $\mathrm{MCG}^*(\mathbb{S}_4^2)$ on the character variety $\chi(\mathbb{S}_4^2)$ or of the group $\mathrm{Aut}[S]$ on S for $S \in Fam$ is one and the same problem.

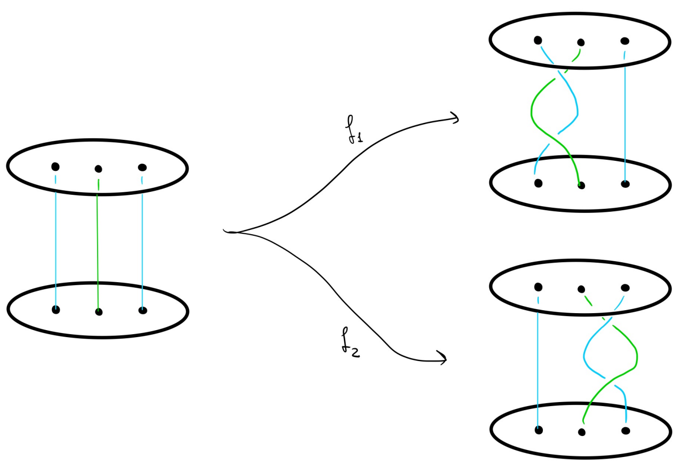
2.4 Recap: A Nice Diagram

The aim of this last section is to summarise the link between the huge amount of groups and subgroups we introduced in this chapter. This will be also the opportunity to better explain an isomorphism (Φ in the diagram below) we have not already seen.

$$
\begin{tikzcd}
\Gamma_2 \arrow[r, "\cong"] & \mathrm{MCG}_0(\mathbb{S}^2_4) \arrow[r, hook] \arrow[ddr, dotted, hook, "\Theta"] & \mathrm{MCG}^*(\mathbb{S}^2_4) \arrow[dd, "{[h]\mapsto h_*}"', "\cong"] \arrow[ddr, dotted, hook] & \Gamma_2^* \arrow[l, hook'] \arrow[dd] \arrow[ddr, "``\cong"'] & \\
\mathbb{F}_2 \arrow[u, "\cong"] \arrow[d, "\cong"'] & & & & \\
\pi_1(\mathbb{S}^2_3, t) \arrow[r] \arrow[uur, "\cong", "\Phi"'] \arrow[rr, bend right, "\Psi"'] & \mathrm{Aut}(\pi_1(\mathbb{S}^2_4)) \arrow[r] & \mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4)) \arrow[r, "\varphi\mapsto f_\varphi"] & \mathrm{Aut}[\chi(\mathbb{S}^2_4)] & \mathrm{Aut}[S_{(A,B,C,D)}] \arrow[l, hook']
\end{tikzcd}
$$

Yellow diagram

We collect in this diagram three isomorphisms that are important because, in the green diagram, we will relate them with $\mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4))$ that is one of the main protagonist of this article.
First of all we notice that the fundamental group of the 3-punctured sphere, in perfect analogy with the 4-punctured sphere, is given by $\pi_1(\mathbb{S}^2_3) \cong \langle \alpha, \beta, \gamma \mid \alpha\beta\gamma = 1 \rangle$ and then it holds the isomorphism with $\mathbb{F}_2$.
Something really similar happens for Γ_2 that we recall can be seen either as the subgroup of matrices $M \in \mathrm{PSL}(2, \mathbb{Z})$ such that $M \equiv I \bmod(2)$, or as the subgroup of the mapping class group that preserve the orientation and stabilises the four punctures. In any case we can see Γ_2 as $\langle s_x, s_y, s_z \mid s_x s_y s_z = 1 \rangle$, therefore the isomorphism with $\mathbb{F}_2$ holds.
A little more subtle is the isomorphism $\mathrm{MCG}_0(\mathbb{S}^2_4) \cong \mathbb{F}_2$, but we have seen it in an example in the end of Section 1.1. For the convenience of the reader we briefly recall that we can see $\mathrm{MCG}_0(\mathbb{S}^2_4) \cong \langle f_1, f_2 \rangle$ where

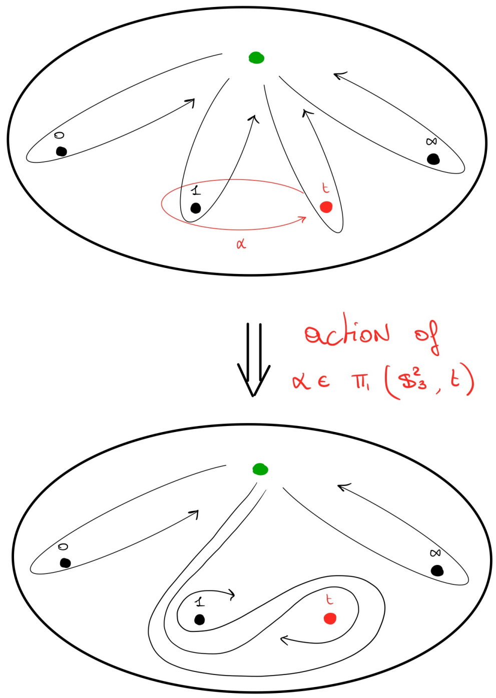


We stress that in this way we can easily build the isomorphism $\pi_1(\mathbb{S}^2_3, t) \cong \mathrm{MCG}_0(\mathbb{S}^2_4)$. To do so we set t as the point at infinity of the 4-punctured sphere, that we can thus imagine as a 3-punctured closed disk. Then we can send the loop around 0 and 1 to f_1 and the loop around 1 and ∞ to f_2.

Green diagram

First of all we would like to investigate the Ψ morphism because it explains the link between Γ_2 and $\mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4))$.

We will consider for simplicity the 4-punctured sphere as $\mathbb{CP}^1 \setminus \{0, 1, t, \infty\}$ for t a complex number different from 0 and 1. There is a natural action of $\pi_1(\mathbb{S}^2_3, t)$ on $\pi_1(\mathbb{S}^2_4)$. Let us explain this with a drawing in a particular case. Let us consider $\alpha \in \pi_1(\mathbb{S}^2_3, t)$ as the loop turning one time around 1. It induce an action by "twisting" the generators $\gamma_1, \gamma_t \in \pi_1(\mathbb{S}^2_4)$ respectively represented by simple loops around 1 and t.



In formulae we have that :

$$\alpha : \begin{cases} \gamma_0 \mapsto \gamma_0 \\ \gamma_1 \mapsto (\gamma_1\gamma_t)\gamma_1(\gamma_1\gamma_t)^{-1} \\ \gamma_t \mapsto (\gamma_1\gamma_t)\gamma_t(\gamma_1\gamma_t)^{-1} \\ \gamma_\infty \mapsto \gamma_\infty \end{cases}$$

In particular we can prove that for any loop in $\pi_1(\mathbb{S}^2_3, t)$, the induced action is a morphism inside $\mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4))$.

Blue and Red diagrams

The punctured version of the DNB theorem establishes the isomorphism $\mathrm{MCG}^*(\mathbb{S}^2_4) \cong \mathrm{Out}^\star(\pi_1(\mathbb{S}^2_4))$. Thanks to this, we can define the action of the extended mapping class group on $\chi(\mathbb{S}^2_4)$. This action is very complicated, but fortunately we have the subgroup Γ_2^* that preserves the punctures of the sphere and that is easier to handle. Two key results play a fundamental role:

- The images of $\mathrm{MCG}^*_0(\mathbb{S}^2_4)$ and of Γ_2^* inside $\mathrm{Aut}[\chi(\mathbb{S}^2_4)]$ coincide.

- Since Γ_2^* preserve the punctures, then its action preserves the surfaces $S_{(A,B,C,D)} \subseteq \chi(\mathbb{S}_4^2)$ and the image of Γ_2^* inside $\mathrm{Aut}[S_{(A,B,C,D)}]$ has index bounded by 24 and for a generic choice of the parameters they are isomorphic.

3 The Symplectic Structure of Closed Character Varieties

A character variety is an object way more rich than a singular algebraic variety. In some particular cases depending on the properties of the Lie group G (for instance, G complex and semisimple), the character variety $\chi(\Gamma, G)$ can be endowed with an extra structure. In particular if Γ is a closed surface group (i.e. the surface S has no punctures) the character variety can be endowed with a symplectic structure, while in the more general case of a non-closed surface group, it carries a more general structure called Poisson structure. We recall that a holomorphic symplectic structure is defined by a global closed non degenerate holomorphic 2-form. A symplectic structure allows to deeply study the manifold: in fact the main advantages it brings are a canonical isomorphism between a tangent space and its dual and the lagrangian foliation. In this chapter we will introduce the symplectic structure in the general case for a non punctured surface S. In particular we will follow the work of Goldman in [4].

3.1 The Construction

The main ingredient in order to build the symplectic structure comes from what we said about the Zarisky tangent space in Section 1.4. In fact we recall that for a smooth point ρ, it holds that

$$T_{[\rho]}\chi(\Gamma, G) = \frac{T_\rho \operatorname{Hom}(\Gamma, G)}{T_\rho \mathcal{O}} = H^1(\Gamma, \mathfrak{g}_\rho).$$

If a variety carries a symplectic form ω, then its dimension is even, say equal to $2n$, and it holds that $\omega^{\wedge 2n}$ is a volume form for the variety. Therefore, before going on, we should ask ourselves if a general character variety is even dimensional, i.e. we should study the dimension of $H^1(\Gamma, \mathfrak{g}_\rho)$. We will not go through the technical details, but we will give an idea to how to compute this dimension.
First of all let us consider the alternating sum (Euler characteristic)

$$\Sigma(\mathfrak{g}_\rho) = \dim H^0(\Gamma, \mathfrak{g}_\rho) - \dim H^1(\Gamma, \mathfrak{g}_\rho) + \dim H^2(\Gamma, \mathfrak{g}_\rho).$$

Lemma 3.1. *$\Sigma(\mathfrak{g}_\rho)$ does not depend on ρ.*

proof. See [4] Section 1.5. the differential), the Euler characteristic $\Sigma(\mathfrak{g}_\rho)$ is independent of ρ. □

Moreover we have the following result.

Lemma 3.2. *If $\Gamma = \pi_1(S)$ for a closed (i.e. non punctured) surface S, then*

$$\Sigma(\mathfrak{g}_\rho) = \Sigma(S) \dim \mathfrak{g} = (2 - 2g_S) \dim \mathfrak{g},$$

where g_S is the genus of S.

Idea. Since we have proven that $\Sigma(\mathfrak{g}_\rho)$ is independent of ρ, we can compute it taking the trivial representation $\tilde{\rho}$. In this case $\mathfrak{g}_{\tilde{\rho}}$ coincide with $\mathfrak{g}$, and then any $H^*(\Gamma, \mathfrak{g})$ is of the form $H^*(\Gamma, \mathbb{C}) \otimes \mathfrak{g}$. By what we said in the end of Section 1.4, it holds the isomorphism $H^*(\Gamma, \mathbb{C}) \cong H^*(S, \mathbb{C})$ and then the thesis follows. □

So we have achieved that

$$\dim H^1(\Gamma, \mathfrak{g}_\rho) = \dim H^0(\Gamma, \mathfrak{g}_\rho) + \dim H^2(\Gamma, \mathfrak{g}_\rho) + (2g_S - 2) \dim \mathfrak{g},$$

and we have then to investigate $\dim H^0(\Gamma, \mathfrak{g}_\rho) + \dim H^2(\Gamma, \mathfrak{g}_\rho)$.

Lemma 3.3. *It holds that*

$$\dim H^0(\Gamma, \mathfrak{g}_\rho) = \dim H^2(\Gamma, \mathfrak{g}_\rho) = \dim Z(\rho).$$

Idea. Via the isomorphism $H^*(\Gamma, \mathfrak{g}_\rho) \cong H^*(S, E_\rho)$ seen in the end of Section 1.4, we can use the Poincaré duality. In particular in our case we have that

$$H^2(\Gamma, \mathfrak{g}_\rho) \cong H^0(\Gamma, \mathfrak{g}_\rho^*)^*.$$

By the fact that we are working with a semisimple Lie algebra, the Killing form B (that is Ad-invariant, symmetric and non-degenerate) allows us to establish an isomorphism $\mathfrak{g}_\rho \cong \mathfrak{g}_\rho^*$ as Γ-modules (and not only as Lie algebras). Moreover, $H^0(\Gamma, \mathfrak{g}_\rho)$ is exactly the Lie algebra of the stabiliser $Z(\rho)$ and the thesis follows. □

Then, finally,

$$\dim H^1(\Gamma, \mathfrak{g}_\rho) = 2 \dim Z(\rho) + (2g_S - 2) \dim \mathfrak{g}$$

that is actually even. We remark also that this quantity does not depend on ρ, until it is a smooth point.

We can then now focus in building the holomorphic symplectic 2-form. In particular what we have to do is to build a family of alternating bilinear maps

$$\omega_\rho \colon H^1(\Gamma, \mathfrak{g}_\rho) \times H^1(\Gamma, \mathfrak{g}_\rho) \to \mathbb{C},$$

holomorphically depending on ρ, and that defines a non degenerate closed 2-form ω. To do so we will use three main ingredients:

- the cup product $\cup \colon H^1(\Gamma, \mathfrak{g}_\rho) \otimes H^1(\Gamma, \mathfrak{g}_\rho) \to H^2(\Gamma, \mathfrak{g}_\rho \otimes \mathfrak{g}_\rho)$,
- the Killing form $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$,
- and the Poincaré isomorphism $\int_S \colon H^2(S, \mathbb{C}) \to \mathbb{C}$.

In particular we can consider their composition

$$H^1(\Gamma, \mathfrak{g}_\rho) \otimes H^1(\Gamma, \mathfrak{g}_\rho) \xrightarrow{\cup} H^2(\Gamma, \mathfrak{g}_\rho \otimes \mathfrak{g}_\rho) \xrightarrow{B} H^2(\Gamma, \mathbb{C}) \cong H^2(S, \mathbb{C}) \xrightarrow{\int_S} \mathbb{C}$$

and see if it actually verify all the conditions we need.

Remark. First of all we notice that B is defined on Lie algebras and not on Γ-modules. That is *a priori* a problem for us. Anyway, the Killing form is Ad-invariant (by trace properties) and then we can consider B as a bilinear form on the Γ-module $\mathfrak{g}_\rho$.

The aim of the whole following part is to show that the function we have defined is actually the symplectic form we were looking for. As we will see it will be hard to prove that the form is closed. To do so we will follow the proof Goldman gave in [4], but we will need to introduce some new elements in order to understand it.

Theorem 3.4. *Let us set, for each $\rho \in \mathrm{Hom}^{irr}(\Gamma, G)$ and for $u, v \in H^1(\Gamma, \mathfrak{g}_\rho)$,*

$$\omega_\rho(u, v) = \int_S B(u \cup v).$$

Then ω is a holomorphic symplectic form.

Proof. Skew-symmetric
Since the cup product is skew-symmetric and the Killing form is symmetric, their composition is skew-symmetric.
Non-degenerate
We recall that we are discussing the case in which G is semisimple, then the Killing form is non degenerate.
Holomorphic
In [4], section 3, it is shown that ω can be seen as an algebraic tensor in the coordinates coming from the algebraic structure on $\mathrm{Hom}(\Gamma, G)$ and clearly it makes it holomorphic.
Closed
As we said, for this part we need some extra work. We then pause the proof for a minute and we introduce a little part of the Riemann-Hilbert theory we need. ◻

3.2 Interlude: The Riemann-Hilbert Correspondence

The best environment for showing that a form is closed is the (twisted) de Rham cohomology of a manifold. In this section we will see how to establish an isomorphism between $H^1(\Gamma, \mathfrak{g}_\rho)$ and $H^1(S, \mathrm{Ad}\, P)$, the first twisted cohomology group of S with values in the flat vector bundle $\mathrm{Ad}\, P$.

We briefly start from the beginning in order to give a complete picture that describes the element we mentioned just before.

Definition. A *flat vector bundle* (∇, V) on the surface S is the data of a couple of a vector bundle $V \to S$ and a flat connection.

Definition. Given a vector bundle V, we define the *V-valued exterior k-differential forms* on S as $\Omega^k(S, V) := \Gamma(S, V \otimes \Omega^k(S)) = \Gamma(S, V) \otimes_{C^\infty(S)} \Omega^k(S)$.

Definition. Given a vector bundle $V \to S$ a *connection* ∇ on V is a differential operator that allows to differentiate sections. We have a global and a local description of a connection:

- if $s \in \Gamma(S, V)$ is a global section, then a connection is an operator $\nabla \colon \Gamma(S, V) \to \Omega^1(S, V)$ that satisfies the following Leibniz rule for $f \in C^\infty(S)$:

$$\nabla(fs) = df \otimes s + f\nabla s.$$

  We recall that $\Gamma(S, V)$ is a $C^\infty(S)$-module.

- if locally (i.e. in coordinates for a trivialising open set $U \subset S$ for V) the section $s_{|U}$ is of the form $\sum_{i=i}^n s_i \otimes e_i$ where $s_i \colon U \to \mathbb{C}$ are smooth maps and e_i are a local basis for $V_{|U} \cong U \times \mathbb{C}^n$, then a connection acts

$$\nabla_{|U}(s_{|U}) = \sum_{i=1}^n \left(ds_i + \sum_{j=1}^n \Omega_i^j s_j \right) \otimes e_i = (d + \Omega)\Big(\sum_{i=i}^n s_i \otimes e_i \Big) = (d + \Omega)(s_{|U})$$

  where $\Omega = (\Omega_j^i)$ is the connection 1-form, alias a $n \times n$ matrix with entries in $\Omega^1(U)$ associated to ∇. In particular we can say that $\Omega \in \Omega^1(U, \mathrm{End}(V_{|U}))$.

An important remark that follows from the definition is to notice that the space of all possible connections on a vector bundle is an affine space modelled on $\Omega^1(S, \mathrm{End}(V))$.

Definition. A connection ∇ is *flat* if it holds that $\nabla^2 = 0$.

In particular if a connection is flat, we can build a cochain complex $\Omega^*(S, V)$ and its cohomology in the classical way.

Remark. In literature is usually used the symbol ∇ only for differentiate sections and the symbol d_∇ to represent the induced exterior derivative of higher orders, i.e. $d_\nabla \colon \Omega^k(S, V) \to \Omega^{k+1}(S, V)$, $k \geq 1$.

Theorem 3.5. *When $\Gamma = \pi_1(S)$, then the cohomology of the complex $\Omega^*(S, V)$ defined as*

$$H^*(S, V) := \frac{\ker d_\nabla}{\mathrm{Im}\, d_\nabla}$$

is isomorphic to the group cohomology $H^*(\Gamma, V)$.

This is a crucial point for us, because it allows to transpose the group cohomology $H^1(\Gamma, \mathfrak{g}_\rho)$ in terms of de Rham cohomology, as desired.
Since in our setting of connection we can not define $\mathfrak{g}_\rho$, we would like to find its equivalent.

Remark. What are we saying, as the title of the section suggests, is part of the Riemann-Hilbert correspondence. It establish (*a posteriori*) a symplectomorphism $\mathrm{Hom}(\Gamma, G)/Inn(G) \cong \mathcal{F}/\mathcal{G}$ or, to be more precise, an equivalence of deformation theories

$$Inn(G) \curvearrowright \mathrm{Hom}(\Gamma, G) \quad \leftrightsquigarrow \quad \mathcal{G} \curvearrowright \mathcal{F}$$

where $\mathcal{G}$ is the gauge group and $\mathcal{F}$ is the space of flat connections. In particular we will see in the next section that the tangent space of the space $\mathcal{F}$ is exactly $Z^1(S, \mathrm{Ad}\, P)$ and the tangent space on the smooth points of the space $\mathcal{F}/\mathcal{G}$ is exactly $H^1(S, \mathrm{Ad}\, P)$, in perfect analogy with what we studied for the representation variety.

In order to build the equivalent of $\mathfrak{g}_\rho$ we will consider the principal G-bundle P_ρ. We recall that the fundamental group Γ of a surface is isomorphic to $\mathrm{Deck}(\tilde{S} \to S)$, the group of deck transformation associated to the universal covering, and that it can act on S. Moreover, via a representation ρ, it can act on G. We pose then

$$\Pi \colon P_\rho := (\tilde{S} \times G)/\Gamma \to S$$

with the action

$$\gamma \cdot (\tilde{s}, g) = (\gamma \cdot \tilde{s}, \rho(\gamma) g).$$

Remark. Usually we see the fibers of a principal bundle as a copy of the Lie group. We stress that the fiber $\Pi^{-1}(s)$ is also isomorphic to the orbit $\mathcal{O}_s$. In fact, since the action is free, it holds the isomorphism

$$\begin{array}{ccc} \mathcal{O}_s & \to & G \\ g \cdot s & \mapsto & g \end{array}$$

It can be shown that $P_\rho \to S$ is a principal fiber bundle endowed with a canonical flat connection D_ρ. We can associate to this principal bundle a vector bundle using the adjoint representation $\mathrm{Ad} \colon G \to GL(\mathfrak{g})$. In particular we define

$$\pi \colon \ \mathrm{Ad}\, P := P \times_{\mathrm{Ad}_\rho} \mathfrak{g} = (P \times \mathfrak{g})/G \to S.$$

The elements of $\operatorname{Ad} P$ are equivalence classes $[p, X] = \{(q, Y) \mid (p, X) = (g \cdot q, \operatorname{Ad}_g Y) \text{ for a } g \in G\}$, in particular we have that the fibers $\pi^{-1}(s)$ are isomorphic to $\mathfrak{g}$ (for the moment only as vector spaces). This comes from the fact that for any element in the equivalent class $[p, X]$ we have that $\Pi(p) = \Pi(q) = s$, and then

$$\pi^{-1}(s) = \{[p, X] \mid \Pi(p) = s \text{ and } X \in \mathfrak{g}\}.$$

This bundle inherits the structure of flat vector bundle via the connection ∇_ρ induced by D_ρ. Moreover we can give the fibers of $\operatorname{Ad} P$ the structure of the Γ-module $\mathfrak{g}_\rho$ and thanks to Theorem 3.5 we can identify $H^1(\Gamma, \mathfrak{g}_\rho)$ with $H^1(S, \operatorname{Ad} P)$.

Now it makes sense to build a differential form on this tangent space in order to build the analogous of the form built on $\operatorname{Hom}(\Gamma, G)/Inn(G)$.

Let η, θ two $\operatorname{Ad} P$-valued 1-forms on S. Their wedge product is a 2-form on S taking values in the $\mathfrak{g} \otimes \mathfrak{g}$-bundle $\operatorname{Ad} P \otimes \operatorname{Ad} P$ associated to P. As before, we can use the Killing form B to build the (standard) 2-form $B(\eta \wedge \theta) \in \Omega^2(S, \mathbb{C})$.

3.3 Closeness

In order to prove that the form we defined in Theorem 3.4 is closed we will change setting and work with the twisted de Rham cohomology of S, with coefficients in the vector bundle $\operatorname{Ad} P$. To do so, we have to build the tangent space of the space of flat connections and build the analogous differential form on it.

Let us denote $\mathcal{A}$ the space of all connections of the principal G-bundle $\Pi \colon P_\rho \to S$.

Lemma 3.6. *$\mathcal{A}$ is an affine space modelled on $\Omega^1(S, \operatorname{Ad} P)$.*

Idea. We will not go through the elements of gauge theory needed for detail the proof, anyway we will give an idea of the result. We recall that a connections of a principal fiber bundle $P \to S$ is a choice of an horizontal subbundle of TP, we recall that the vertical subbundle is fixed, it is the tangent space to the orbits and its fibers are then isomorphic to $\mathfrak{g}$. In particular to give a connection is equivalent to give a 1-form with values in $\mathfrak{g}$ whose kernel is the horizontal subbundle representing the connection. In this way, a connection is well defined only if its form is Ad-invariant and horizontal, i.e. an element of $\Omega^1(S, \operatorname{Ad} P)$, as desired. □

We know that the space $\mathcal{F}$ of flat connections is given by the set of ∇ such that $d_\nabla^2 = 0$. In order to find the tangent space to the space of flat connection we can differentiate this expression with respect to a tangent vector $\eta \in \Omega^1(S, \operatorname{Ad} P)$.

$$\lim_{t \to 0} \frac{(d_\nabla)^2 - (d_\nabla + t\eta)^2}{t} = \lim_{t \to 0} \frac{d_\nabla^2 - d_\nabla^2 - t d_\nabla \eta - t^2 \eta \wedge \eta}{t} = -d_\nabla \eta$$

and it is zero exactly when $\eta \in Z^1(S, \operatorname{Ad} P)$, that then represent the tangent space. We have not entered the details of the gauge group action, but it is possible to show that the space $B^1(S, \operatorname{Ad} P)$ is the tangent space to the orbit under its action.

Now it makes sense to build on this tangent space the analogous of the differential form we had for $\operatorname{Hom}(\Gamma, G)//Inn(G)$.

Let η, θ two $\operatorname{Ad} P$-valued 1-forms on S. Their wedge product is a 2-form on S taking values in the $\mathfrak{g} \otimes \mathfrak{g}$-bundle $\operatorname{Ad} P \otimes \operatorname{Ad} P$ associated to P. As before, we can use the Killing form B to build the (non-twisted) 2-form $B(\eta \wedge \theta) \in \Omega^2(S, \mathbb{C})$. Therefore we can take its integral and get the alternating pairing

$$\omega \colon (\eta, \theta) \mapsto \int_S B(\eta \wedge \theta) \tag{3}$$

on $\Omega^1(S, \operatorname{Ad} P)$, that means a differential form on $\mathcal{A}$.

Theorem 3.7. *The form (3) is closed and reduces to a symplectic form on $\mathcal{F}/\mathcal{G}$.*

Proof. Since the expression of the form does not depend on the point $\nabla \in \mathcal{A}$, it is invariant under translation and it is then closed. In particular it descend to a closed form on $\mathcal{F}$, but it is degenerate. If suffice to show that the annihilator of this form is exactly $B^1(S, \operatorname{Ad} P)$. From the Leibnitz rule it follows that for any connection ∇, section $\sigma \in \Omega^0(S, \operatorname{Ad} P)$ and 1-form θ it holds that

$$dB(\sigma \wedge \theta) = B(d_\nabla \sigma \wedge \theta) + B(\sigma \wedge d_\nabla \theta).$$

Let us then consider $\eta = d_\nabla \sigma \in B^1(S, \operatorname{Ad} P)$ and $\theta \in Z^1(S, \operatorname{Ad} P)$. Then

$$\int_S B(d_\nabla \sigma \wedge \theta) = \int_S dB(\sigma \wedge \theta) + \int_S B(\sigma \wedge \cancel{d_\nabla \theta}) = 0,$$

because of Stokes theorem. Therefore $B^1(S, \operatorname{Ad} P) \subseteq \ker \omega$. For the other inclusion, we set $\eta \in \ker \omega$, that means that for any θ it holds that $\int_S B(\eta \wedge \theta) = 0$. In particular, since S is a compact closed surface, we have that $\int_S \colon H^2(S, \mathbb{C}) \to \mathbb{C}$ is an isomorphism and then $B(\eta \wedge \theta) \in B^2(S, \mathbb{C})$. We have to show that this implies that $\eta \in B^1(S, \operatorname{Ad} P)$. In particular we know that, for $G = \mathrm{SL}(2, \mathbb{C})$, $B(\eta \wedge \theta) = 4 \operatorname{tr}(\eta \wedge \theta)$ and therefore

$$\operatorname{tr}(\eta \wedge \theta) = d\omega$$

$$\operatorname{tr} \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & -\eta_{11} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & -\theta_{11} \end{pmatrix} = d\omega$$

$$2\eta_{11} \wedge \theta_{11} + \eta_{12} \wedge \theta_{21} + \eta_{21} \wedge \theta_{12} = d\omega$$

And since θ is closed and the wedge product on the first cohomology group is non degenerate, η must be exact as desired. ◻

Therefore we lead to the conclusion of the proof of Theorem 3.4. We recall, as we said in the previous section, that the Riemann-Hilbert correspondence is actually a symplectomorphism. We summarise here the analogies the Riemann-Hilbert correspondence establish.

$$\begin{array}{ccc}
\operatorname{Hom}(\gamma, G) & \leftrightsquigarrow & \mathcal{F} \\[1em]
Inn(G) \curvearrowright \operatorname{Hom}(\Gamma, G) & \leftrightsquigarrow & \mathcal{G} \curvearrowright \mathcal{F} \\[1em]
T_\rho \operatorname{Hom}(\gamma, G) \cong Z^1(\Gamma, \mathfrak{g}_\rho) & \leftrightsquigarrow & T_\nabla \mathcal{F} \cong Z^1(S, \operatorname{Ad} P) \\[1em]
T_\rho \mathcal{O}^{Inn(G)}(\rho) \cong B^1(\Gamma, \mathfrak{g}_\rho) & \leftrightsquigarrow & T_\nabla \mathcal{O}^{\mathcal{G}}(\nabla) \cong B^1(S, \operatorname{Ad} P) \\[1em]
H^1(\Gamma, G) & \leftrightsquigarrow & H^1(S, \operatorname{Ad} P)
\end{array}$$

In particular for a closed (i.e. non punctured) complex surface S we built a complex symplectic structure on its character variety.

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