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Complex Geometry through the Eyes of a Torus

A complex (s)tori

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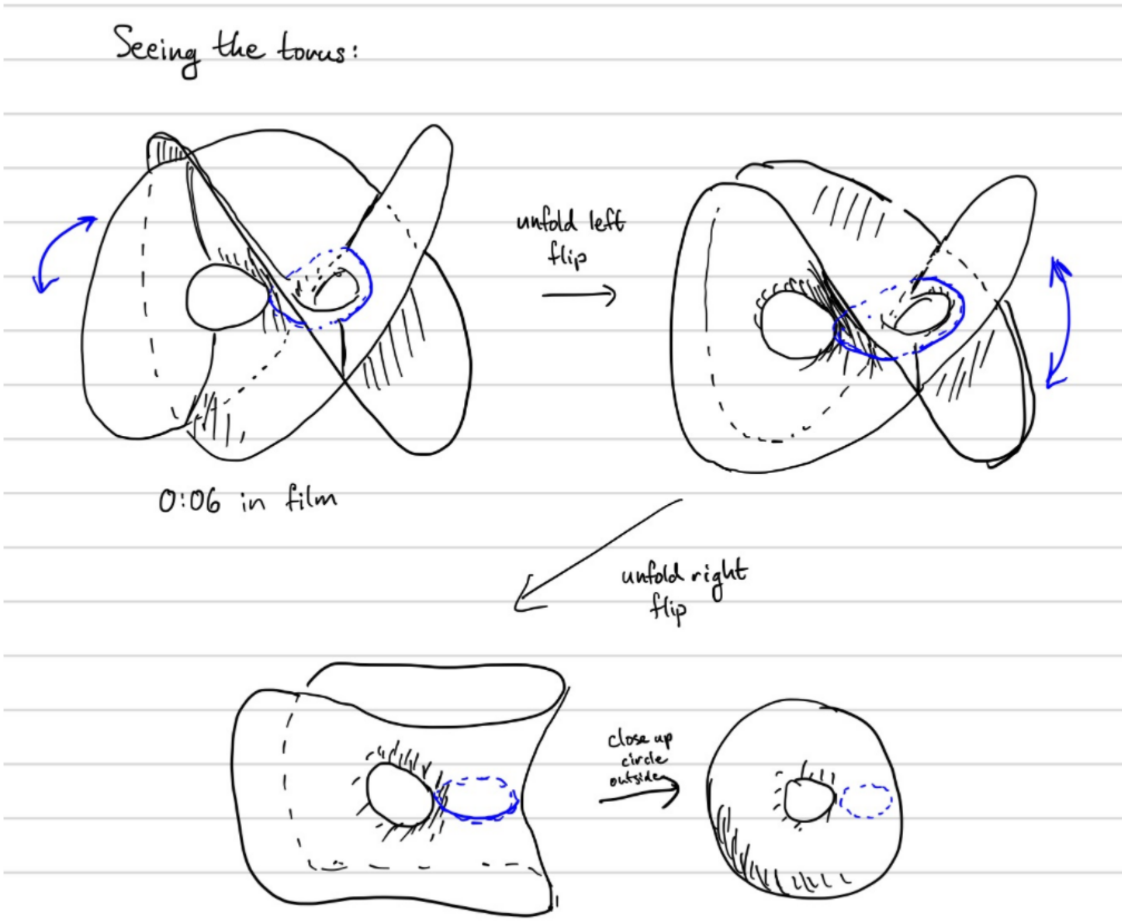


Website: Visualizing complex points of elliptic curves

Introduction

Mathematicians gave birth to complex geometry in 19th century. It is the result of the interaction of complex analysis, differential geometry and algebraic geometry that were very prolific fields in those years. If we were somewhere in ancient Greek mythology, the genealogical tree would be similar to the following:

$$\begin{tikzcd}
\text{Physics} \arrow[dd] & \text{Analysis} \arrow[d] \arrow[dr] & \text{Number Theory} \arrow[dr] & \text{Commutative Algebra} \arrow[d] \\
& \text{Diff Geom} \arrow[dl] \arrow[d] & \text{Cpx Analysis} \arrow[dl] & \text{Alg Geom} \arrow[dll] \arrow[ddl] \\
\text{Riem Geom} \arrow[dr] & \text{Cpx Geom} \arrow[d] \arrow[dr] & & \\
& \text{Kähler Geom} \arrow[d] & \text{Cpx Alg Geom} \arrow[dl] & \\
& \text{Hyperkähler Geom} & &
\end{tikzcd}$$

The main topic of this work are complex tori. One may wonder why among all possible manifolds one should waste time in studying just a unique class of examples. The answer to this question can be found hidden between the lines of this thesis, but here we collect some interesting motivations.
First of all the topological concept of torus arises almost in every mathematical field mentioned in the above diagram: the configuration space of a double pendulum is a torus, the topological torus can be endowed with a (unique!) differential structure, on any field one can define algebraic tori, even dimensional tori can be endowed with a complex structure (far away from being unique!), any complex torus is Kähler, some Kähler tori are complex projective algebraic varieties and so on.
Secondly, we focus our attention on the fact that all differential structures on a torus are diffeomorphic, while there is an infinite number of non isomorphic complex structures on a even dimensional torus. This special feature allows us to realise how the notion of ”isomorphism” is more rigid in the holomorphic context than in the smooth context. The most incredible thing is to see how manifolds that appear to be ”the same” from all other points of view may have such different behaviour when studied through a complex lens: some tori are projective, some tori are not, some tori allow meromorphic functions, some do not, very few tori admit a principal polarization, a lot of them do not.
Besides being interesting in itself, it offers a pretext to introduce, to study and fully understand some standard tools used in complex geometry such as line bundles, divisors, Kähler forms.
The first nine sections follow the aim to introduce complex geometry through the eyes of a complex torus. The last four sections on the other hand try to describe the many faces and behaviours that different complex structures on a torus may assume. We will in particular distinguish between projective tori (which are called *abelian varieties*) and non projective ones, observing that the large majority of tori falls in this second class. Finally, we will study what happens when we consider polarized tori, i.e. abelian varieties endowed with an integer Kähler form and we will find the moduli space of principally polarized abelian varieties.
In the last section we deal with a new and original problem: we study Hyperkähler structures on

complex marked tori, trying to build an explicit moduli space. We will prove (for now only in complex dimension 2) that the *extremal volumes* (TP2) of the homology classes of an Hyperkähler 2-tours with respect to three complex structures allow us to reconstruct locally the starting torus, giving an immersion of the moduli space of Hyperkähler structures on a marked 2-torus in $\mathbb{R}^{18}$. Further developing of this theory may be to study if the immersion is also an embedding and if we can endow the moduli space, that is real 12-dimensional, with some analytic, Kähler or Hyperkähler structures. Clearly once achieved something in the 2-dimensional case one may wonder if it holds for higher dimensions.

Contents

1 Basic Definitions and Results

Definition. Let V be a vector space of complex dimension g and $\Gamma \subseteq V$ a lattice (i.e. an abelian discrete subgroup of rank 2g). We define a complex torus as $X = V/\Gamma$ where the action is given by the translation by elements $\gamma \in \Gamma$.

We denote by $\pi\colon V \longrightarrow X$ the universal covering map. We would like to give X a complex structure: let $\Omega \subseteq V$ be an open set such that for all $\gamma \in \Gamma \setminus \{0\}$

$$(\Omega + \gamma) \cap \Omega = \emptyset.$$

Since $\pi_1(X) = Deck(\pi) = \{f\colon V \longrightarrow V \ s.t.\ \pi \circ f = \pi\} \cong \Gamma$ is the set of the translations by elements of Γ, Ω is a fundamental open set for the covering. This implies that $\pi_{|\Omega}$ is a homeomorphism and we can give X a complex structure such that $\pi_{|\Omega}$ is a biholomorphism. Let us build an atlas as follows: let A_1, A_2 be open sets in X and let us denote by φ_i the biholomorphism $(\pi^{-1})_{|A_i}\colon A_i \longrightarrow B_i$, where $B_i \subseteq V$ is one of the fundamental open sets in $\pi^{-1}(A_i)$. If the intersection is non empty, we set $A_{12} = A_1 \cap A_2$. The change of coordinates is given by $\varphi_1 \circ \varphi_2^{-1} := \varphi_{12}\colon \pi^{-1}(A_{12}) \longrightarrow \pi^{-1}(A_{12})$ and hence $\varphi_{12} \circ \pi = \pi$, so we have that φ_{12}, restricted to each connected component of $\pi^{-1}(A_{12})$, is a translation for some element in Γ and thus a biholomorphism, as desired.

In order to study function on holomorphic tori we need the following definition.

Definition. A function $f\colon U \subseteq X \longrightarrow \mathbb{C}$ is holomorphic (meromorphic) if and only if $f \circ \pi\colon V \longrightarrow \mathbb{C}$ is holomorphic (meromorphic).

Thus global holomorphic maps on a complex torus are the holomorphic maps on V that are Γ-periodic, hence constant for Liouville theorem. Notice that it is consistent with the general fact $\mathcal{O}(X) = \mathbb{C}$ for all connected compact complex manifolds.
Now we focus our attention on holomorphic maps between tori. Next proposition will be very useful because it gives a very concrete description of these maps. There are two distinguished types of holomorphic maps between complex tori: homomorphisms and translation.

Definition. Let $X = V/\Gamma$ and $X' = V'/\Gamma'$ be complex tori of dimension g and g'. A homomorphism from X to X' is a holomorphic map $f\colon X \longrightarrow X'$, compatible with the group structures.

Definition. The translation by an element $x_0 \in X$ is the holomorphic map $\tau_{x_0}\colon x \mapsto x + x_0$

Proposition 1.1. *Let $h\colon X \longrightarrow X'$ be a holomorphic map.*

(1) There is a unique homomorphism $f\colon X \longrightarrow X'$ such that $h = \tau_{h(0)} \circ f$.

(2) There is a unique $\mathbb{C}$-linear map $F\colon V \longrightarrow V'$ with $F(\Gamma) \subseteq \Gamma'$ inducing the homomorphism f.

Proof. Define $f = \tau_{-h(0)}h$ and consider the composed map $f \circ \pi\colon V \longrightarrow X'$. We can lift it to a holomorphic map $F\colon V \longrightarrow V'$

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[dr, "f\circ\pi"'] & V' \arrow[d, "\pi'"] \\ & X' \end{tikzcd}$$

such that $F(0) = 0$. Since the diagram commutes, $F(v + \gamma) - F(v) \in \Gamma'$. Thus the continous map $v \mapsto F(v + \gamma) - F(v)$ is constant and $F(v + \gamma) = F(v) + F(\gamma)$. This implies the derivatives of F are periodic and hence constant by Liouville theorem. This implies that F is linear. The uniqueness follows by covering theory. $\square$

Corollary 1.2. *Group operations on a complex torus are holomorphic, hence every complex torus is a complex Lie group.*

Remark. The map $\pi\colon V \to X$ is the standard exponential map in Lie groups theory, $\exp\colon \mathfrak{g} \to G$.

Definition. The additive group of the homomorphisms between two complex tori X and X' is denoted by $Hom(X, X')$.

Remark. We have two important faithful representations of the homomorphism group:

- The analytic representation:

$$\rho_a\colon Hom(X, X') \to Hom_{\mathbb{C}}(V, V').$$

- The rational representation

$$\rho_r\colon Hom(X, X') \to Hom_{\mathbb{Z}}(\Gamma, \Gamma') \cong \mathbb{Z}^{4gg'}.$$

Corollary 1.3. $Hom(X, X') \cong \mathbb{Z}^m$ *for some* $m < 4gg'$.

In $Hom(X, X')$ there are special maps called isogenies. We will use them to talk about the dual complex torus and the abelian varieties.

Definition. An isogeny from a complex torus X to a complex torus X' is a surjective homomorphism $f\colon X \to X'$ with finite kernel. Its degree is the cardinality of $\ker f$.

Remark. A homomorphism $X \to X'$ is an isogeny if and only if it is surjective and $dimX = dimX'$.

Let $X = V/\Gamma$ be a complex torus and let Γ' a lattice of V such that $\Gamma' \subseteq \Gamma$. The projection $X' \to X$ is an isogeny. It is clearly surjective and of finite kernel: indeed $v + \Gamma' \mapsto \Gamma$ if and only if $v \in \Gamma \setminus \Gamma'$ which corresponds to a finite number of point in X. Conversely, all isogenies are of this type.

Proposition 1.4. *To be isogenic is an equivalence relation.*

Proof. The only non obvious assertion to prove is that this relation is symmetric. If $f\colon \to V'/\Gamma'$ is an isogeny, hence $F(\Gamma) \subseteq \Gamma'$ is a finite index subgroup. It follows that there exists a natural number n such that $n\Gamma' \subseteq F(\Gamma)$, therefore nF^{-1} induces an isogeny in the opposite direction. $\square$

2 Elliptic Curves

Before studying complex tori in several dimension, let us consider the case $g = 1$. It will turn out to be slightly different from all other cases and in section 12.2 the main difference is pointed out.

Definition. A complex torus of complex dimension 1 is said elliptic curve.

In this section we will find a moduli space that parametrizes complex structures over elliptic curves and we will prove that every elliptic curve is biholomorphic to a smooth cubic in $\mathbb{CP}^2$. Moreover we start introducing theta functions and divisors in this simpler case.

Definition. Two lattices Γ and Γ' are said equivalent if and only if $\mathbb{C}/\Gamma \cong \mathbb{C}/\Gamma'$

We will use matrices in $GL(2,\mathbb{R})$ to represent lattices. In fact, given a invertible matrix M, we associate to it the lattice generated by its columns. We want to find conditions that makes two lattices equivalent: first of all we have that, if we fix a lattice, its bases are parametrized by $GL(2,\mathbb{Z})$. Therefore all possible lattices up to basis are parameterised by $GL(2,\mathbb{R})/GL(2,\mathbb{Z})$.

Proposition 2.1. *Two different lattices $\Gamma_1, \Gamma_2 \subseteq \mathbb{C}$ are equivalent if and only if exists a complex number $a \in \mathbb{C}^*$ such that $a\Gamma_1 = \Gamma_2$.*

Proof. Clearly, if $a\Gamma_1 = \Gamma_2$, the tori X_1 and X_2 are isomorphic because the multiplication map $a\colon \mathbb{C} \to \mathbb{C}$ descends to an isomorphism. On the other hand if $f\colon X_1 \to X_2$ is an isomorphism (in particular a group morphism) it is represented by a function $F\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto az$ such that $a\Gamma_1 = \Gamma_2$. $\square$

Hence the moduli space of complex structures on a torus of complex dimension 1 is:

$$GL(1,\mathbb{C})\backslash GL(2,\mathbb{R})/GL(2,\mathbb{Z}).$$

A geometric interpretation of this fact arises from an explicit description of the basis of the lattice. Let $\Gamma = \langle \tau_1, \tau_2 \rangle_{\mathbb{Z}}$ be a lattice: we want to build all equivalent lattices acting on τ_1 and τ_2 via $GL(1,\mathbb{C})$ and $GL(2,\mathbb{Z})$. Let us suppose $|\tau_1| \leq |\tau_2|$, there exists $a \in \mathbb{C}^*$ such that $\langle a\tau_1, a\tau_2 \rangle = \langle 1, \tau \rangle$ and up to a change of sign of τ (that obviously does not change the lattice and that is represented by a matrix in $GL(2,\mathbb{Z})$) we can suppose $Im(\tau) > 0$ and hence $\tau \in \mathbb{H}$, the upper half plane. Moreover, by construction, we have $|\tau| \geq 1$. Up to now we acted by $GL(1,\mathbb{C})$ and by a change of sign in $GL(2,\mathbb{Z})$, so we just have to act via $SL(2,\mathbb{Z})$. We can rephrase the action of $SL(2,\mathbb{Z})$ on the lattice as an action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ as shown in this diagram, where $A \in SL(2,\mathbb{Z})$ is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{tikzcd} \tau \arrow[r] \arrow[d, "A"'] & \langle 1, \tau \rangle \arrow[d, "A"] \\ \tau' = \dfrac{a\tau + b}{c\tau + d} \arrow[r] & \langle c\tau + d, a\tau + b \rangle \end{tikzcd}$$

Under this action the moduli space of elliptic curve is

$$SL_2(\mathbb{Z})\backslash\mathbb{H} = \{\tau \in \mathbb{H} \mid |\tau| \geq 1 \text{ and } |Re\tau| \leq 1/2\}.$$

For a complete and exhaustive proof we refer to (Hai), Proposition 1.18.

2.1 Meromorphic functions

Now we focus our attention on meromorphic functions on a elliptic curve parametrized by $(1,\tau)$. We will see that in higher dimension meromorphic functions on complex tori may have a very different behaviours depending on the complex structure (i.e. depending on the lattice).

Definition. A theta function is defined as $\theta(z) = \sum f_n(z)$, where $f_n(z) = e^{\pi i(n^2\tau+2nz)}$ for all $z \in \mathbb{C}$.

The definition is well posed because the series converges uniformly on every compact set in $\mathbb{C}$ due to the Weierstrass M-test. Since f_n are holomorphic, θ is holomorphic.

Proposition 2.2. *The following equalities holds*

- $\theta(z+1) = \theta(z)$.
- $\theta(z+\tau) = e^{-i\pi(\tau+2z)}\theta(z)$.
- $\theta(\frac{\tau}{2}+\frac{1}{2}) = 0$ *and the zero is simple.*

Definition. A translated theta function is $\theta^x(z) := \theta(z - \frac{\tau}{2} - \frac{1}{2} - x)$

A translated theta function has simple zeros in $z \in \{x + \Gamma\}$.
We can now define a family of meromorphic functions:

$$R(z) = \frac{\prod_{i=1}^{m} \theta^{x_i}(z)}{\prod_{i=1}^{n} \theta^{y_i}(z)}.$$

Form the proposition one can check that:

- $R(z+1) = R(z)$.
- $R(z+\tau) = (-1)^{m+n} e^{-2\pi i[(m-n)z+\sum y_i - \sum x_j]} R(z)$

Hence R is double periodic if and only if $m = n$ and $\sum y_i - \sum x_j \in \mathbb{Z}$. If these conditions are satisfied R descends to a meromorphic function on X.

Before going on, let us recall some important theorems about holomorphic and meromorphic functions on Riemann surfaces.

Proposition 2.3. *Let $f\colon M \to N$ be a holomorphic map between Riemann surfaces.*

- *If M is compact, then f is proper and so it is a branched covering map. Furthermore, f is surjective, so N is also compact.*
- *If N is compact and f is proper, then M is also compact.*

Proof. It is an easy consequence of the open mapping theorem. □

Proposition 2.4. *Any meromorphic function $f\colon M \to \mathbb{C}$ has an unique extension to a holomorphic map $f\colon M \to \mathbb{CP}^1$.*

As a consequence of this two results we can prove the following.

Theorem 2.5. *Let M be a compact Riemann surface (we are interested in the case in wich M is a elliptic curve) and f a meromorphic function on M with only one pole, then M is biholomorphic to $\mathbb{CP}^1$.*

Proof. From 2.4 we know that $f\colon M \to \mathbb{CP}^1$ is a holomorphic map and from 2.3 it is a covering map of degree 1, hence a biholomorphism since $\mathbb{CP}^1$ is simply connected. $\square$

We can now prove the following fundamental result about classification of meromorphic functions on elliptic curves.

Proposition 2.6. *Let p_i and q_i points on X and let $x_i \in \pi^{-1}(p_i)$ and $y_i \in \pi^{-1}(q_i)$. Does exist a meromorphic function on X with zeros in p_i and poles in q_i if and only if $\sum x_i - \sum y_i \in \Gamma$ (i.e. $\sum p_i - \sum q_i = 0 \in X$).*

Proof. Let $\Gamma = \langle 1, \tau \rangle$ and assume $\sum x_i - \sum y_i = n + m\tau$. We can substitute x_1 with $x_1 - n\tau$ and nothing changes because $\pi(x_1) = \pi(x_1 - n\tau) = p_1$. Now $\sum x_i - \sum y_i = n + m\tau \in \mathbb{Z}$ and the assertion is proved. Conversely, assume there exists a meromorphic function on $\mathbb{C}/\Gamma$ with the given zeroes and poles. Lifting it to $\mathbb{C}$, we obtain a double periodic function f with zeroes in $x_i + \Gamma$ and poles in $y_i + \Gamma$. By contradiction, assume that $\sum x_i - \sum y_i = w \notin \Gamma$. Choose x_0 and y_0 such that $x_0 - y_0 = w$. Now $x_0 + \sum x_i - y_0 - \sum y_i = 0 \in \mathbb{Z}$ and hence it descends to a rational theta function g. The quotient map g/f is doubly periodic and descends to a meromorphic function on $\mathbb{C}/\Gamma$ with a single zero in $\pi(x_0)$ and a single pole in $\pi(y_0)$, but this generates a biholomorphism between the elliptic curve and $\mathbb{CP}1$ that is a contradiction. $\square$

2.2 Divisors

The study of divisors is very related to meromorphic functions on a complex manifold: let us introduce this fundamental tool in the case of elliptic curves. This will be fundamental for enhancing Riemann-Roch theorem and prove that every elliptic curve is biholomorphic to a smooth cubic in $\mathbb{CP}^2$, a very strong result since we will see that in higher dimensions the general complex torus is not projective.

Definition. A (Weil) divisor is a function $D\colon M \to \mathbb{Z}$ with discrete support: i.e. $D = \sum n_i p_i - \sum m_j q_j$, with $n_i, m_j \geq 0$ and $n_i, m_j \neq 0$ for finitely many indices.

This relates with the study of meromorphic functions because we can define this sequence

$$\begin{array}{ccccc} \mathcal{M}^* & \xrightarrow{div} & Div(M) & \xrightarrow{deg} & \mathbb{Z} \\ f & \longmapsto & div(f) & & \\ & & D & \longmapsto & \sum n_i - \sum m_j \end{array}$$

in which div is the function that associates to every meromorphic function the formal sum of zeroes and poles counted with their multiplicity and deg associates to every divisor the explicit sum of the coefficients. From the theory of meromorphic functions we can notice that the composition is null, i.e. $deg(div(f)) = 0$.

Definition. A principal divisor is $D = div(f)$ for some $f \in \mathcal{M}^*$.

Let us denote with $Div^0(M)$ the kernel of deg. We have that

$$\{\text{principal divisors}\} \subseteq Div^0(M) \subseteq Div(M)$$

Definition. The Picard group of M is

$$Pic(M) = \frac{Div(M)}{\{\text{principal divisors}\}}$$

The definition is well posed because principal divisors (and $Div^0(M)$ as well) forms a normal subgroup with the operations

- $div(fg) = div(f) + div(g)$
- $div(1/f) = -div(f)$.

If we denote with $Pic^0(M)$ the kernel of the quotient degree map $deg\colon Pic(M) \to \mathbb{Z}$, we can show that

Theorem 2.7. *If M is a elliptic curve, the morphism*

$$\varphi\colon M \to Pic^0(M)$$
$$p \mapsto \overline{[p] - [0]}$$

is a group isomorphism, where the overline stands for the equivalent class in Pic.

Proof. See (Deb) Theorem 3.1. □

Remark. It holds that

$$Pic^0(X) \cong \frac{Div^0(X)}{\{\text{principal divisors}\}}$$

2.3 Riemann-Roch and consequences

The Riemann-Roch theorem is a classical result in complex geometry and it encodes and put in relations topological data, divisors and meromorphic functions. We enunciate two slightly different versions: the first is very general and holds for every Riemann surface, while the second is specific for elliptic curves.

Definition. We set $\mathcal{O}_D := \{f \in \mathcal{M}$ such that $div(f) + D \geq 0\}$, where "$\geq$" means that any non null coefficients must be positive.

This space can be endowed with the structure of vector space with respect to the usual additive structure. Note that $0 \in \mathcal{O}_D$ setting $div(f \equiv 0) = +\infty$. It can be endowed with the more general structure of sheaf of vector spaces on M. Let us recall this general fact about 1-forms on Riemann surfaces:

Proposition 2.8. *Let α be a meromorphic $(1,0)$-form on a compact Riemann surface of genus g. Then the number of zeros minus the number of poles counted with their molteplicity is $2g-2$.*

Proof. It follows from the Poincaré-Hopf theorem. □

Theorem 2.9 (Riemann-Roch, version 1, (TP1))**.** *Let M be a compact Riemann surface of genus g and let D be a divisor. It holds that*

$$dimH^0(M, \mathcal{O}_D) - dimH^1(M, \mathcal{O}_D) = 1 + d - g.$$

Theorem 2.10 (Riemann-Roch, version 2, (BL))**.** *Let M be a elliptic curve and let D be a divisor with $deg(D) = d > 0$. It holds that*

$$dimH^0(M, O_D) = d$$

Proof. The genus of an elliptic curve is $g = 1$, so the RHS of the equation is just d. On the LSH, thanks to Serre duality for compact Riemann surfaces, we have that

$$dimH^1(M, O_D) = dimH^1(\Omega^0_D) = dimH^0(\Omega^1_{-D}) = dim(\Omega^1_{-D}) = 0$$

because $-D$ has degree $-d$ and for the proposition 2.8 can not exist a 1-form α such that $div(\alpha)-D \geq 0$ since for an elliptic curve $2g - 2 = 0$. □

The following theorem is a very deep result. It states that any complex torus of dimension 1 is biholomorphic to an algebraic subset of $\mathbb{CP}^2$. It may not appear as incredible as it is, but we will see in next sections that it does not hold in higher dimension. Since we will use it in the proof, we recall Hurwitz's formula.

Theorem 2.11 (Hurwitz's formula)**.** *Let $f : M \to N$ a map between compact Riemann surfaces and let g_M, g_N denote the genera. The Euler characteristic of M and N must satisfy the following relation:*

$$2 - 2g_M = d(2 - 2g_N) - \sum_{p \in M} (m_p(f) - 1)$$

where d is the degree of f and $m_p(f)$ is the multiplicity of f in p.

Proof. See (TP1) Theorem 5.3. □

Theorem 2.12. *Every holomorphic structure over an elliptic curve M is biholomorphic to a smooth cubic in $\mathbb{CP}^2$.*

Proof. Let p be a point of M and set $D = \{2p\}$. Riemann-Roch theorem imples that $dim(O_D) = 2$,so there must exist at least one meromorphic function in $\mathcal{O}_D$. If it had a simple pole in p, M would be $\mathbb{CP}^1$, so the pole must be double and the function would provide a 2:1 branched covering over $\mathbb{CP}^1$. Hurwitz formula implies that there exist 4 branch points in $\mathbb{CP}^1$, one of which is ∞. Let p_i denote the other three branch points in $\mathbb{C}$. Let us consider the algebraic curve $w = (z - p_1)(z - p_2)(z - p_3)$ that defines a smooth cubic in $\mathbb{CP}^2$ and a projection map $[z, w, u] \to [z, u] \in \mathbb{CP}^1$. Let us now compare M with this cubic from the viewpoint of Riemann existence theorem: both surfaces map onto $\mathbb{CP}^1$ with the same branch points, for both the covering degree is 2 and the monodromy exchanges the branches. Since there is a unique monodromy action that achieves this, by Riemann existence theorem we can conclude that M and the cubic are biholomorphic. □

3 Complex vector spaces and complex manifolds

The first step of our journey through the geometry of complex tori is their topology, in particular their (co)homology. Before starting, we recall some results about differential forms over (open sets of) complex vector spaces. We will present just the proofs that are strictly related to the contents of next sections.

3.1 Complex vector spaces and multilinear forms

Definition. A real vector space V of real dimension N endowed with an endomorphism $I\colon V \to V$ such that $I^2 = -id$ is called almost complex vector space, and I is called almost complex structure of V.

Notice that $I \in GL_N(V)$ and an almost complex structure allows us to say that (V, I) is a complex vector space because I corresponds to the multiplication by $i \in \mathbb{C}$: i.e. $(a + ib)v := av + bI(v)$, thus for vector spaces the notions of almost complex structure and complex structure coincide. This implies that $N = 2n$.
For a real vector space V, the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ is denoted by $V_{\mathbb{C}}$ and there is a canonical identification of V with the subset $V \otimes 1 \subseteq V_{\mathbb{C}}$. V is the subset that is left invariant under conjugation: $\overline{v \otimes \lambda} = v \otimes \overline{\lambda}$.

Definition. Let I be a complex structure on a real vector space V and let (with an abuse of notation) $I\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$ its $\mathbb{C}$-linear extension. We set the eigenspaces

$$V^{1,0} = \{v \in V_{\mathbb{C}}\colon I(v) = i \cdot v\} := V \quad \text{and} \quad V^{0,1} = \{v \in V_{\mathbb{C}}\colon I(v) = -i \cdot v\} := \overline{V}$$

Notice that $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ under the map $v \mapsto \frac{1}{2}(v - iI(v)) \oplus \frac{1}{2}(v + iI(v))$.

Remark. One should be aware of the existence of two almost complex structure. One given by I and the other given by i. They coincide on the subspace $V^{1,0}$, but differ by a sign on $V^{0,1}$.

In order to study multilinear forms on a complex vector space, we focus our attention on complex structures on the dual vector space.

Proposition 3.1. *Let V be a real vector space endowed with an almost complex structure I. Then the dual space $V^* := Hom_{\mathbb{R}}(V, \mathbb{R})$ has a natural almost complex structure given by $I(f)(v) = f(I(v))$. The induced decomposition on $(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$ is given by:*

$$(V^*)^{1,0} = \{f \in Hom_{\mathbb{R}}(V, \mathbb{C})\colon f(Iv) = if(v)\} = (V^{1,0})^* = Hom_{\mathbb{C}}(V, \mathbb{C}) := V^*$$

$$(V^*)^{0,1} = \{f \in Hom_{\mathbb{R}}(V, \mathbb{C})\colon f(Iv) = -if(v)\} = (V^{0,1})^* = Hom_{\overline{\mathbb{C}}}(V, \mathbb{C}) := \overline{V}^*$$

In the sequel we will always regard $V_{\mathbb{C}}$ as the complex vector space with respect to i.

Definition. A function $f \in Hom_{\mathbb{C}}(V, \mathbb{C})$ is $\mathbb{C}$-linear, in the sense that for any $\lambda \in \mathbb{C}$, $f(\lambda v) = \lambda f(v)$.
A function $f \in Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$ is $\mathbb{C}$-antilinear, in the sense that for any $\lambda \in \mathbb{C}$, $f(\lambda v) = \overline{\lambda} f(v)$.

This construction generalizes to multilinear forms: let V be a complex vector space of complex dimension n and let $r \leq n$. We denote with $\Lambda^r V_{\mathbb{C}}^*$ the complex vector space of r-multilinear alternating forms over V. Its basis is $\{\alpha_I\}_I$ where I is a multiindex of length r, $\alpha_I := \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r}$ and $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $V^* := Hom_{\mathbb{C}}(V, \mathbb{C})$. Therefore his complex dimension is $\binom{n}{r}$. It holds that:

$$\Lambda^r V^*_{\mathbb{C}} = \bigoplus_{p+q=r} \Lambda^{p,q} V^*$$

where $\Lambda^{p,q} V^* := \Lambda^p V^* \oplus_{\mathbb{C}} \Lambda^q \overline{V}^*$ is generated by $\{\alpha_I \wedge \overline{\alpha}_J\}_{I,J}$ with $|I| = p$ and $|J| = q$.
We would like to link the notion of complex structure with the notion of multilinear forms.

Definition. An almost complex structure I on V is compatible with the scalar product $\langle \cdot, \cdot \rangle$, if $\langle I(v), I(w) \rangle = \langle v, w \rangle$.

Definition. The fundamental form associated to $(V, \langle \cdot, \cdot \rangle, I)$ is the form

$$\omega := \omega(v, w) := -\langle v, I(w) \rangle = \langle I(v), w \rangle$$

Lemma 3.2. *The fundamental form is real and of type (1,1), i.e.* $\omega \in \Lambda^2 V^* \cap \Lambda^{1,1} V^*$.

Proof. The operator I satisfies

$$\langle v, I(w) \rangle = -\langle w, I(v) \rangle$$

hence ω is an alternating 2-form, i.e. $\omega \in \Lambda^2 V^*_{\mathbb{C}}$. Moreover, since

$$I(\omega)(v, w) := \omega(I(v), I(w)) = \langle I^2(v), I(w) \rangle = \omega(v, w)$$

one finds $I(\omega) = \omega$, that is $\omega \in \Lambda^{1,1} V^*$. □

Regarding the complex structure I as the multiplication for i, we can prove the following.

Lemma 3.3. *The form* $H = H(u, v) := \omega(u, iv) - i\omega(u, v)$ *is a positive Hermitian form on* $(V, \langle \cdot, \cdot \rangle)$.

Proof. First of all let us notice that $H(v, v) = \langle v, v \rangle - i\omega(v, v) = \langle v, v \rangle$. So $H(v, v) > 0$ for all $v \neq 0$. By a straightforward calculation one gets $H(u, v) = \overline{H(v, u)}$ since the form ω is of type $(1, 1)$ and real. Finally we have to prove that H is $\mathbb{C}$-linear:

$$\begin{aligned} H(iu, v) &= \omega(iu, iv) - i\omega(iu, v) = \\ &= \omega(u, v) - i\omega(-u, iv) = \\ &= \omega(u, v) + i\omega(u, iv) = \\ &= i(-i\omega(u, v) + \omega(u, iv) = \\ &= iH(u, v). \end{aligned}$$

□

3.2 Differential forms on a complex vector space

Definition. Let $U \subseteq V$ be an open set. A smooth function $\omega \colon U \to \Lambda^{p,q} V^*$ is said (p, q)-differential form over U.

A smooth function $f\colon U \to \mathbb{C}$ is holomorphic if and only if its differential is $\mathbb{C}$-linear, i.e. if $df \in \Lambda^{1,0}V^*$, i.e. Cauchy-Riemann equations hold. This allows us to generalize the Cauchy-Riemann equations for functions of several variables and we care about this because if $F\colon A \subseteq V \to B \subseteq W$ is holomorphic, then his differential is $\mathbb{C}$-linear and the pull-back F^* preserves the type of the form. This is very important because it proves that the type of a form is globally well-defined over a complex manifold since the change of coordinates are holomorphic.

Definition. We will denote by $\Omega^{p,q}(U)$ the space of (p,q)-differential forms on U.

Proposition 3.4. *There is a natural decomposition:*

$$\Omega^r(U) = \bigoplus_{p+q=r} \Omega^{p,q}(U)$$

Definition. Let $\mathrm{d}\colon \Omega^r(U) \to \Omega^{r+1}(U)$ the complex linear extension of the usual exterior differential. Then we can define two other differential operators, namely ∂ and $\overline{\partial}$ that acts on $\Omega^{p,q}$

$$\partial\colon \Omega^{p,q}(U) \to \Omega^{p+1,q}(U) \text{ and } \bar{\partial}\colon \Omega^{p,q}(U) \to \Omega^{p,q+1}(U).$$

They are defined by $\partial := \Pi^{p+1,q} \circ d$ and $\overline{\partial} := \Pi^{p,q+1} \circ d$, where $\Pi^{p,q}\colon \Omega^r(U) \to \Omega^{p,q}(U)$ is the projection and $r = p + q$.

Proposition 3.5. *For the differential operators ∂ and $\bar{\partial}$ one has:*

- $\mathrm{d} = \partial + \bar{\partial}$
- $\partial^2 = \bar{\partial}^2 = 0$ *and* $\partial\bar{\partial} = -\bar{\partial}\partial$
- *Leibniz rule is satisfied.*

Since ∂ and $\bar{\partial}$ share the usual properties of the exterior differential d and reflect the holomorphicity of functions, it seems natural to build up a holomorphic analogue of the de Rham complex. As we work here exclusively in the local context, only the local aspects will be discussed. Of course, locally the de Rham complex is exact due to the standard Poincaré lemma. We will show that this still holds true for $\bar{\partial}$ (and ∂).

$$\begin{tikzcd}
\Omega^0(U) \arrow[d, "\mathrm{d}"] \\
\Omega^1(U) \arrow[d, "\mathrm{d}"] \\
\Omega^2(U) \arrow[d, "\mathrm{d}"] \\
\cdots \arrow[d, "\mathrm{d}"] \\
\Omega^n(U) \arrow[d, "\mathrm{d}"] \\
\Omega^{n+1}(U) \arrow[d, "\mathrm{d}"] \\
\cdots \arrow[d, "\mathrm{d}"] \\
\Omega^{2n-1}(U) \arrow[d, "\mathrm{d}"] \\
\Omega^{2n}(U)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & \Omega^0(U) \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & & & \\
 & & & \Omega^{1,0}(U) \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & \Omega^{0,1}(U) \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & & \\
 & & \Omega^{2,0}(U) \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & \Omega^{1,1}(U) \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & \Omega^{0,2}(U) \arrow[dl, "\partial"'] \arrow[dr, "\bar{\partial}"] & & \\
 & \cdots \arrow[dl, "\partial"'] & & \cdots \arrow[dr, "\bar{\partial}"] & & \cdots \arrow[dl, "\partial"'] & & \cdots \arrow[dr, "\bar{\partial}"] & \\
\Omega^{n,0}(U) \arrow[dr] & & & & \Omega^{\frac{n}{2},\frac{n}{2}}(U) \arrow[dl] \arrow[dr] & & & & \Omega^{0,n}(U) \arrow[dl] \\
 & \Omega^{n,1}(U) \arrow[dr] & & \Omega^{\frac{n}{2}+1,\frac{n}{2}}(U) \arrow[dl] \arrow[dr] & & \Omega^{\frac{n}{2},\frac{n}{2}+1}(U) \arrow[dl] \arrow[dr] & & \Omega^{1,n}(U) \arrow[dl] & \\
 & & \cdots \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & & \\
 & & & \Omega^{n,n-1}(U) \arrow[dr] & & \Omega^{n-1,n}(U) \arrow[dl] & & & \\
 & & & & \Omega^{n,n}(U) & & & &
\end{tikzcd}$$

Proposition 3.6 ($\bar{\partial}$-Poincaré lemma)**.** *Let U be an open neighbourhood of the closure of an open ball $B_\epsilon \subset \overline{B}_\epsilon \subset U \subset \mathbb{C}^n$. If $\alpha \in \Omega^{p,q}(U)$ is $\bar{\partial}$-closed and $q > 0$, then there exists a form $\beta \in \Omega^{p,q-1}(B_\epsilon)$ such that $\alpha = \bar{\partial}\beta$ on B_ϵ.*

Proof. See (Huy), proposition 1.3.8. □

We would like to look at the interaction of differential forms and Riemannian metrics: let $U \subset \mathbb{C}^n$ and g a Riemannian metric on U. We say that the metric g is compatible with the almost complex structure I on U if for any $x \in U$ the induced scalar product g_x on T_xU is compatible with the induced complex structure in the sense of the previous section. Also in this case we may define a $(1,1)$-form $\omega \in \Omega^{1,1}(U) \cap \Omega^2(U)$, defined by $\omega(u,v) := g(I(u),v)$, which is called fundamental form of g. Moreover $H := g - i\omega$ defines a positive Hermitian form on the complex vector space (T_xU, g_x).

3.3 Complex structures, differential forms and Dolbeault cohomology

In the very first section we have built complex tori as complex manifolds (i.e. we have given a torus a holomorphic atlas). However, in order to study complex differential forms, we will need an another approach to complex manifolds: we will define an almost complex structure I on a real smooth manifold X and we will find conditions on this structure to make (X, I) a complex manifold. When it happens we will say that the almost complex structure is integrable. This change of perspective makes sense thanks to the following result.

Theorem 3.7 (Newlander-Nieremberg)**.** *Any integrable almost complex structure is induced by a complex structure.*

Thus, complex manifolds and differentiable manifolds endowed with an integrable almost complex structure describe the same geometrical object.

Definition. An almost complex manifold X is a differentiable manifold X together with a vector bundle endomorphism, called almost complex structure:

$$I\colon TX \to TX, \ \text{ with } I^2 = -id.$$

Here TX is the real tangent bundle of the underlying real manifold.

Remark. Given a holomorphic atlas $(U_\alpha, \varphi_\alpha)$ on X, one can define an almost complex structure on $\varphi_\alpha(U_\alpha)$ as in section 3.1. It does not depend neither on the chart, nor on the atlas in the equivalent class.

Proposition 3.8. *Let X be an almost complex manifold. There exist a direct sum decomposition*

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$$

of complex vector bundles on X, such that the $\mathbb{C}$-linear extension on I acts as multiplication by i on $T^{1,0}X$ and as multiplication by $-i$ on $T^{0,1}X$.
Moreover, if X is a complex manifold, then $T^{1,0}X$ is naturally isomorphic (as a complex vector bundle) to the holomorphic tangent bundle $\mathcal{T}X$.

Proof. See (Huy) Proposition 2.6.4. □

As we did on complex vector spaces, we would like to study the dual case and the space of differential forms.

Definition. For an almost complex manifold X one defines the complex vector bundles

$$\Lambda^k_{\mathbb{C}}X := \Lambda^k(T_{\mathbb{C}}X) \quad \text{and} \quad \Lambda^{p,q}X := \Lambda^p(T^{1,0}X)^* \otimes_{\mathbb{C}} \Lambda^q(T^{0,1}X)^*.$$

Proposition 3.9. *There exists a natural direct sum decomposition*

$$\Lambda^k_{\mathbb{C}}X = \bigoplus_{p+q=k} \Lambda^{p,q}X$$

Let us denote with $\Omega^k X$, and $\Omega^{p,q}X$ the spaces of sections. As for complex vector space we have the the morphisms d, ∂ and $\bar{\partial}$.
We will now give four definitions of integrable complex structure, for a proof of their equivalence see (Huy), prop 2.6.15 et seq.

Definition. Let (X, I) be an almost complex manifold. Then I is called integrable and (X, I) complex manifold if one of the following equivalent conditions holds

- $d\alpha = \partial\alpha + \bar{\partial}\alpha$ for all $\alpha \in \Omega^k X$.
- If $\alpha \in \Omega^{1,0}X$ then $d\alpha$ has no pieces of type $(0,2)$.
- The Lie bracket of vector fields preserve $T^{0,1}X$, i.e. the distribution $T^{0,1}X$ is involutive and hence integrable.
- $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$

We conclude this section giving the definition of Dolbeault cohomology.

Definition. Let X be endowed with an integrable almost complex structure (i.e. a complex structure). Then the (p,q)-Dolbeault cohomology group is the vector space

$$H^{p,q}(X) := \frac{Ker(\bar{\partial}\colon \Omega^{p,q}X \to \Omega^{p,q+1}X)}{Im(\bar{\partial}\colon \Omega^{p,q-1}X \to \Omega^{p,q}X)}.$$

4 Cohomology of complex tori

We will see that complex tori may have very different behaviours depending on their complex structure. However from the C^∞-point of view they are all diffeomorphic (and hence homeomorphic), thus topological data encoded in the C^∞-structure, as the De Rham cohomology, will be the same for any torus. Some informations about the holomorphic structure is encoded in the Dolbeault cohomology, in particular in the Hodge decomposition.

4.1 Singular Cohomology and De Rham Cohomology

We will deduce the main result by two different ways, the firs one, from (BL), is more abstract and precise. The second one, from (Deb) is more constructive and simple.
Recall that a torus X, as a real manifold, is diffeomorphic to a product of $2g$ circumferences. More precisely, if $X = V/\Gamma$ with $g = dim_{\mathbb{C}}(V)$ and $\{\gamma_1, \dots, \gamma_{2g}\}$ is a $\mathbb{Z}$-basis for Γ, we have that

$$X \cong \bigtimes_{i=1}^{2g} \frac{\gamma_i \mathbb{R}}{\gamma_i \mathbb{Z}} := (S^1)^{2g}.$$

We want to compute $H_1(X, \mathbb{Z})$, because, as we will see, all the cohomology groups of the torus can be computed out of it. Notice that, according to the basic definition we gave, $\pi_1(X) \cong Deck(\pi) \cong \Gamma$. Since Γ is an abelian subgroup of $\mathbb{C}^g$, it follows that $H_1(X, \mathbb{Z}) \cong \Gamma$ by the standard fact that $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$. Hence, according to universal coefficient theorem, we have:

$$H^1(X, \mathbb{Z}) \cong Hom\Big(H_1(X, \mathbb{Z}), \mathbb{Z}\Big) \cong Hom(\Gamma, \mathbb{Z}).$$

Proposition 4.1 (Künneth Formula)**.** *Let X and Y be topological spaces. We have the following exact sequence:*

$$0 \to \bigoplus_{i+j=k} \Big[H_i(X, \mathbb{Z}) \otimes H_j(Y, \mathbb{Z})\Big] \to H_k(X \times Y, \mathbb{Z}) \to \bigoplus_{i+j=k} Tor_1^{\mathbb{Z}}\Big(H_i(X, \mathbb{Z}) \otimes H_j(Y, \mathbb{Z})\Big) \to 0.$$

In our case $X = Y = S^1$ and $\bigoplus_{i+j=k} Tor_1^{\mathbb{Z}}\Big(H_i(X, \mathbb{Z}) \otimes H_j(Y, \mathbb{Z})\Big) = 0$, so the first map is an isomorphism. We can now compute the higher cohomology groups.

Lemma 4.2. *The canonical map $\Lambda^r H^1(X, \mathbb{Z}) \to H^r(X, \mathbb{Z})$ induced by the cap product is an isomorphism for all $r > 0$.*

Proof. It is a consequence of the compatibility of Künneth's formula with the cup product. For details see (BL) Lemma 1.3.1. □

This is really important because now we have that:

$$\Lambda^r H^1(X, \mathbb{Z}) \cong \Lambda^r Hom(\Gamma, \mathbb{Z}) := Alt^r(\Gamma, \mathbb{Z}) \cong H^r(X, \mathbb{Z}).$$

Where $Alt^r(\Gamma, \mathbb{Z})$ denotes the group of alternating r-forms on Γ with values on $\mathbb{Z}$.
Thanks to universal coefficient theorem

$$H^r(X, \mathbb{C}) \cong H^r(X, \mathbb{Z}) \otimes \mathbb{C},$$

and it is a general result that

$$Alt^r(\Gamma, \mathbb{Z}) \otimes \mathbb{C} \cong Alt^r_{\mathbb{R}}(V, \mathbb{C}) = \Lambda^r V^*_{\mathbb{C}}.$$

Theorem 4.3. *If $X = V/\Gamma$ is a complex torus, its De Rham cohomology is given by the following isomorphisms for all $r > 0$*

$$H^r_{DR}(X) \cong \Lambda^r V^*_{\mathbb{C}}$$

Proof. Putting together all the previous result we have

$$H^r_{DR}(X) \cong H^r(X,\mathbb{C}) \cong H^r(X,\mathbb{Z}) \otimes \mathbb{C} \cong Alt^r(\Gamma,\mathbb{Z}) \otimes \mathbb{C} \cong Alt^r_{\mathbb{R}}(V,\mathbb{C}) = \Lambda^r V^*_{\mathbb{C}}$$

□

We would like to give a more explicit construction of this isomorphism: notice that if X is a complex torus and $\omega \in \Omega^r(X)$, then $\pi^*\omega \in \Omega^r(V)$ and it is invariant for the translation by an element $\gamma \in \Gamma$: in fact, since $\pi \circ \tau_\gamma = \pi$,

$$(\pi \circ \tau_\gamma)^*\omega = \tau^*_\gamma \circ \pi^*\omega = \pi^*\omega$$

because the pullback is a controvariant functor.
Locally a form $\eta \in \Omega^r(V)$ is of the form $\eta = \sum \eta_I dx^I$, and it is Γ-periodic (i.e. invariant for all τ^*_γ) if and only if η_I are Γ-periodic.

Lemma 4.4. *Let ω be a closed r-form on X and let $\tau(x) = x + a$ the translation for an element $a \in X$. The form $\tau^*\omega - \omega$ is exact.*

Proof. Let $X = V/\Gamma$ and $\tau(x) = x + a$ on V. For all $x, x_1, \dots, x_r$ in V we set

$$\eta_a(x)(x_1,\dots,x_{r-1}) = \int_0^1 \omega(x+ta)(a,x_1,\dots,x_{r-1})dt,$$

where we look at ω as a Γ-periodic r-form on V. This implies that η_a is Γ-periodic too. We have to prove that $d\eta_a(x) = \omega(x+a) - \omega(x)$. Since ω is closed

$$0 = d\omega(x+ta)(a,x_1,\dots,x_r) = (\omega'(x+ta)\cdot a)(x_1,\dots,x_r) - \sum_{j=1}^r (-1)^{j-1}(\omega'(x+ta)\cdot x_j)(a,x_1,\dots,\hat{x}_j,\dots,x_r).$$

Moreover

$$\begin{aligned} d\eta_a(x)(x_1,\dots,x_r) &= \sum_{j=1}^r (-1)^{j-1}(\eta'_a(x)\cdot x_j)(x_1,\dots,\hat{x}_j,\dots,x_r) = \\ &= \sum_{j=1}^r (-1)^{j-1}\int_0^1 (\omega'(x+ta)\cdot x_j)(a,x_1,\dots,\hat{x}_j,\dots,x_r)dt. \end{aligned}$$

Putting together these results the thesis follows:

$$d\eta_a(x)(x_1,\dots,x_r) = \int_0^1 (\omega'(x+ta)\cdot a)(x_1,\dots,x_r)dt = [\omega(x+ta)(x_1,\dots,x_r)]_0^1 = \omega(x+a) - \omega(x).$$

□

By this lemma we will prove directly that every differential form on a complex torus is in the same cohomology class of a constant form, that is exactly $H^r_{DR}(X) \cong \Lambda^r V^*_{\mathbb{C}}$.

Definition. Let Π a plurirectangle with sides a basis of Γ. For all function $g\colon V \to \mathbb{C}$, we define is average $\tilde{g}\colon V \to \mathbb{C}$ as

$$\tilde{g}(z) := \frac{\int_\Pi g(z+y)dy}{\int_\Pi dy}$$

Remark. If g is Γ-periodic, $\tilde{g}$ is constant.

Definition. If $\omega = \sum \omega_I dx^I$ is a differential form on a torus, we define its average as $\tilde{\omega} := \sum \tilde{\omega}_I dx^I$.

Theorem 4.5. *Every closed differential form $\omega \in \Omega^r(X)$ is in the same cohomology class of a constant form.*

Proof.

$$\begin{aligned}\tilde{\omega}(z) - \omega(z) &= \frac{\int_\Pi [\tau_{-y}^* \omega(z+y) - \omega(z)]dy}{\int_\Pi dy} \overset{\text{Lemma4.4}}{=} \\ &= \frac{\int_\Pi d\eta(z)dy}{\int_\Pi dy} = \\ &= d\left(\frac{\int_\Pi \eta(z)dy}{\int_\Pi dy}\right)\end{aligned}$$

□

Hence all differential forms on a complex torus are cohomologue to a constant form. We will see that all real closed (1,1)-forms positive definite are cohomologue to a constant form with the same properties.

Proposition 4.6. *Let $X = V/\Gamma$ a complex torus. The morphism $\Lambda^r V_{\mathbb{C}}^* \to H_{DR}^r(X)$ is bijective.*

Proof. We have already seen that it is surjective. let us prove injectivity: let $(\gamma_1, \dots, \gamma_{2g})$ be a basis for Γ. The integral of a constant form $\sum \omega_{j_1 \dots j_r} dx_1 \wedge \dots \wedge dx_r$ over the image in X of the hypercube with edges the elements $\gamma_{j_1}, \dots, \gamma_{j_r}$ is $\omega_{j_1 \dots j_r}$. Hence a constant form is exact (i.e. its integral is zero over all the possible choices of $\gamma_{j_1}, \dots, \gamma_{j_r}$) if and only if it is the null form. □

4.2 Dolbeault Cohomology

It is a standard fact that holomorphic r-forms over a complex manifold can be endowed with the structure of a sheaf. Let us denote it as Ω_X^r. We enounce the main theorem of this section. We will not give a complete proof, but we will follow the main idea. For all details we refer to (BL), Theorem 1.4.1.

Theorem 4.7. *Let X be a complex torus.*

- *For all $r \geq 0$,*

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^q(\Omega_X^p).$$

- *For every pair (p, q) exists a natural isomorphism*

$$H^q(\Omega_X^p) \cong \Lambda^p V^* \otimes \Lambda^q \overline{V}^*.$$

The main point of all the proof is that complex tori are complex Lie groups. We can therefore see the vector space V as its Lie algebra, i.e. $V = T_0X$ and this allows us to relate constant covectors to differential forms: an element $\varphi \in \Lambda^r V^*$ extends to a holomorphic r-form $(\omega_\varphi)_z := \tau_{-z}^*\varphi$. It is clear that ω_φ is translation-invariant.

Definition. An invariant r-form on X is a differential form $\omega \in \Omega^r(X)$ such that, for all translation τ for an element of X, $\tau^*\omega = \omega$. We denote with $IF^r(X)$ the vector space of invariant r-forms on X.

Lemma 4.8. *Ω_X^r is a free $\mathscr{O}_X$-module of rank $\binom{g}{r}$*

Proof. We define a sheaf morphism

$$\begin{aligned} \Lambda^r V^* \otimes_{\mathbb{C}} \mathscr{O}_X &\to \Omega_X^r \\ \varphi \otimes f &\longmapsto f\omega_\varphi \end{aligned}$$

that is actually an isomorphism because it is an isomorphism on the fibres since each fibre of Ω_X^r is generated by a basis of $\Lambda^r V^*$. □

Remark. Since for all $\varphi \in \Lambda^r V^*$ the differential form ω_φ is invariant, if we take $\varphi = dx^i$, we have that $\omega_\varphi = dx^i$ is invariant (with an obvious abuse of notation).

Remark. The following isomorphisms hold.

$$H^r(X, \mathbb{C}) \cong \Lambda^r V_{\mathbb{C}}^* \cong IF^n(X)$$

Also for the invariant forms there is the canonical splitting $IF^r(X) = \bigoplus_{p+q=r} IF^{p,q}(X)$.

Remark. It holds that

- $IF^{p,0}(X) \cong \Lambda^p V^*$
- $IF^{0,q}(X) \cong \Lambda^q \overline{V}^*$

and hence $\Lambda^p V^* \otimes \Lambda^q \overline{V}^* \cong IF^{p,q}(X)$, where the isomorphism is given by $\varphi_1 \otimes \varphi_2 \mapsto \omega_{\varphi_1 \otimes \varphi_2} := \omega_{\varphi_1} \wedge \omega_{\varphi_2}$

Putting together these results we have proved the following.

Lemma 4.9. *There are natural isomorphisms*

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} IF^{p,q}(X) \cong \bigoplus_{p+q=r} \Lambda^p V^* \otimes \Lambda^q \overline{V}^*.$$

Half of the work is done: it remains to study $H^q(\Omega_X^p)$. For a general fact about Dolbeault cohomology theory, there is a group isomorphism $H^q(\Omega_X^p) \cong H^{p,q}(X)$. In the next sections we will have to deal with Kähler forms that are real closed (1,1)-forms positive definite, hence the comprehension of these groups will be important.
In conclusion, in order to complete the proof of the theorem, it remains to show that $H^{p,q}(X) \cong IF^{p,q}(X)$, but we will not go through this fact. If we assume the isomorphism holds, it would follow that

$$H^r(X, \mathbb{C}) \cong \bigoplus_{p+q=r} IF^{p,q}(X) \cong \bigoplus_{p+q=r} H^{p,q}(X) \cong \bigoplus_{p+q=r} H^q(\Omega_X^p),$$

and

$$H^q(\Omega_X^p) \cong H^{p,q}(X) \cong IF^{p,q}(X) \cong \Lambda^p V^* \otimes \Lambda^q \overline{V}^*.$$

This result will turn out to be very important in the study of Kähler geometry of complex tori because it shows that all real closed (1,1)-forms positive definite are cohomologue to a constant form with the same properties.

Remark. The proof of the isomorphism $H^{p,q}(X) \cong IF^{p,q}(X)$ is based to the fact that both are isomorphic to the group of harmonic (p,q)-forms and involves Hodge theory.

5 Digression: Kähler Forms and The Projective Space

Before going through the geometrical description of complex tori let us recall some important facts concerning Kähler (compact) manifolds and the projective space.

Definition. Let X be a complex manifold. A Kähler structure on X is the choice of a closed positive real $(1,1)$-form ω on X. In this case we call ω a Kähler form on X.

Proposition 5.1. *The set of all Kähler forms on a compact complex manifold X is a convex cone in the linear space $\{\omega \in \Omega^{1,1}(X) \cap \Omega^2(X) \mid d\omega = 0\}$.*

If we denote by I the complex structure of X, we can associate to every Kähler structure a Riemannian metric g and a Hermitian metric H on X:

$$g(\cdot,\cdot) := \omega(\cdot, I(\cdot)) \quad \text{and} \quad H(\cdot,\cdot) := g(\cdot,\cdot) - i\omega(\cdot,\cdot) = \omega(\cdot, I(\cdot)) - i\omega(\cdot,\cdot).$$

It is clear that is sufficient to have two of I, g and ω to uniquely determine the remaining one.

Example 5.1 (The Fubini-Study form)**.** Let $\mathbb{P}^n = \bigcup_{i=1}^n U_i$ the standard open cover and $\varphi_i\colon (z_0, \dots, z_n) \in U_i \mapsto (\frac{z_0}{z_i}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{z_i}) \in \mathbb{C}^n$. A differential form ω on $\mathbb{P}^n$ is a differential form on $\mathbb{C}^n \setminus \{0\}$ that is invariant under the action of $\mathbb{C}^*$: if $\omega = \sum \omega_{IJ} dz_I \wedge d\overline{z}_J$ is a (p,q)-form on $\mathbb{C}^n$ and $t \in \mathbb{C}^*$, ω is invariant if for all I, J

$$\omega_{IJ}(tz) = \frac{1}{t^p \overline{t}^q} \omega_{IJ}(z),$$

in fact in this case

$$t \cdot \omega = t \cdot \sum \omega_{IJ}(z) dz_I \wedge d\overline{z}_J = \sum \omega_{IJ}(tz)\, d(tz)_I \wedge d\overline{(tz)}_J = \frac{t^p \overline{t}^q}{t^p \overline{t}^q} \sum \omega_{IJ} dz_I \wedge d\overline{z}_J = \omega.$$

Let us defines the following local differential forms

$$\omega_i := \frac{i}{2\pi} \partial\overline{\partial} \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right) \in \Omega^{1,1}(U_i).$$

They are clearly invariant under the action of $\mathbb{C}^*$. We want to show that $\omega_i = \omega_j$ on $U_i \cap U_j$ that means that the collection $\{\omega_i\}$ glues to a global form, namely ω_{FS}. Indeed

$$\log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right) = \log \left(\left| \frac{z_j}{z_i} \right|^2 \sum_{k=0}^n \left| \frac{z_k}{z_j} \right|^2 \right) = \log \left(\left| \frac{z_j}{z_i} \right|^2 \right) + \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_j} \right|^2 \right)$$

and $\partial\overline{\partial} \log \left(\left| \frac{z_j}{z_i} \right|^2 \right) = 0$ since $\frac{z_j}{z_i}$ is the j-th coordinate function on U_i

$$\partial\overline{\partial} \log |z|^2 = \partial \Big(\frac{1}{z\overline{z}} \overline{\partial}(z\overline{z}) \Big) = \partial \Big(\frac{z d\overline{z}}{z\overline{z}} \Big) = \partial \Big(\frac{d\overline{z}}{\overline{z}} \Big) = 0.$$

We now check that $\omega_{FS} \in \Omega^{1,1}(\mathbb{P}^n)$

- ω_{FS} is a real (1,1)-form. It is obviously a (1,1)-form and we have to show $\omega_i = \overline{\omega}_i$. Indeed

$$\overline{\frac{i}{2\pi} \partial\overline{\partial} \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right)} = -\frac{i}{2\pi} (-\partial\overline{\partial}) \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right) = \frac{i}{2\pi} \partial\overline{\partial} \log \left(\sum_{k=0}^n \left| \frac{z_k}{z_i} \right|^2 \right).$$

- ω_{FS} is ∂-closed and holomorphic. Indeed $\partial\omega_i = \frac{i}{2\pi}\partial^2\overline{\partial}\log(...) = 0$ and the same happens with $\overline{\partial}$ if we recall that $\partial\overline{\partial} = -\overline{\partial}\partial$.

- ω_{FS} is positive definite. We will not prove it.

Hence ω_{FS} is the Kähler form associated to a metric. We want to prove the most important tool of this Kähler form:

$$\int_{\mathbb{P}^1} \omega_{FS} = 1$$

i.e. it is an integer Kähler form (in the sense of the next definition). Indeed, if $F\colon z \mapsto (1, z, 0, \ldots, 0)$ is a parametrization of a projective line, one has

$$F^*\omega_{FS}(z) = \frac{i}{2\pi}\Big(\frac{dz \wedge d\overline{z}}{(1+|z|^2)^2}\Big).$$

Hence

$$\int_{\mathbb{P}^1} \omega_{FS} = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{dz \wedge d\overline{z}}{(1+|z|^2)^2} = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{dxdy}{(1+x^2+y^2)^2} = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^{\infty} \frac{r}{(1+r^2)^2} dr = 1$$

Moreover, since $\mathbb{P}^1 \cong S^2$, $H^2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$ and hence $[\omega_{FS}] \in H^2(\mathbb{P}^1, \mathbb{Z})$ is a generator.

Definition. Let M be a projective space or a complex torus, a closed r-form ω is integer if for all closed and oriented submanifolds Z of dimension r, the integral $\int_Z \omega \in \mathbb{Z}$.

Remark. Working on a torus allows us to make some definitions simpler: in fact the "correct" definition of integer closed r-form should be formulated integrating on cycles in $H_r(M, \mathbb{Z})$ and not just on submanifolds. On a torus the notions coincide. Moreover we recall that on a torus, that topologically is product of $2g$ circumferences, homology and cohomology groups have no torsion.

Remark. Thanks to the Stokes theorem, the notion of integrality only depends by the cohomology class of ω in $H^r_{DR}(X)$. Moreover we recall that, by the De Rham theorem, $H^r_{DR}(X) \cong H^r(M, \mathbb{R})$ and by universal coefficient theorem and the previous remark, $H^r(M, \mathbb{R}) \cong H^r(M, \mathbb{Z}) \otimes \mathbb{R}$. Therefore integer k-forms are in the set $H^r(M, \mathbb{Z})$ that is precisely the dual of $H_r(M, \mathbb{Z})$.

Theorem 5.2. *If X is a compact complex manifold that is a submanifold of a projective space, there exists on X a closed differential (1,1)-form that is positive definite and integer.*

Proof. The Fubini-Study form meets all the requirements and its restriction on X also does. □

Theorem 5.3. *Any complex torus is Kähler.*

Proof. Any closed, real, positive invariant form of type (1,1) on the vector space V descends to a form on X with the same properties. In particular the set of all Kähler form is non empty for any complex torus. □

In particular any Kähler form is in $\Omega^{1,1}(X) \cap \Omega^2(X)$ and they do not form a subvector space, but only a convex cone (see (Huy) Corollary 3.1.8). In terms of cohomology classes, any Kähler form on a torus defines an element in $H^2_{DR}(X) \cap IF^{1,1}(X)$. We stress that two different Kähler form may be in the same class, in the sense that if $[\omega_1] = [\omega_2]$ it does not imply that they induce the same Kähler structure on X. However Kähler forms in the same cohomology class present some similarities, such as integrality. Thanks to the remark below Lemma 4.9 and the previous remark all possible classes of integer Kähler forms on a complex torus are in the set

$$H^2(X,\mathbb{Z}) \cap H^{1,1}(X) \cong H^2(X,\mathbb{Z}) \cap IF^{1,1}(X).$$

Since the first set is discrete, it may happen that the intersection contains just the null form and it means that there are not integer Kähler forms on the complex torus X. In Theorem 9.2 we will see a necessary and sufficient condition for X to have an integer Kähler form. The naïve idea behind this fact is that $IF^{1,1}(X)$ strongly depends on the lattice Γ and very few lattices admit $(1,1)$-invariant integer forms.

We wonder what happens in case $g = 1$, i.e. when we are working with an elliptic curve $X = \mathbb{C}/\Gamma$. In this case, for dimensional reason $H^2(X,\mathbb{C}) \cong IF^{1,1}(X) \cong \mathbb{C}$ and $H^2(X,\mathbb{Z}) \cong \Gamma$, hence, for any elliptic curve, $H^2(X,\mathbb{Z}) \cap IF^{1,1}(X) = H^2(X,\mathbb{Z}) \neq \{0\}$ and it is consistent with the fact that any elliptic curve is projective.

6 Divisors

We would like to extend to complex tori of higher dimension the theory studied for elliptic curves. It leads some interesting difficulties because the zero locus of meromorphic functions on a complex manifold is a submanifold of codimension 1 (that is just a set of points in the case of elliptic curves). Since the theory we will study applies to any complex compact connected manifold, for this section, we will forget to work on complex tori.

6.1 Weil Divisors

Weil divisors are the most concrete description of a divisor and hence they are very useful to understand concepts that involves divisors. The price we pay is that their definition is too concrete and it is very hard to use it in proofs.

Definition. Let X be a compact complex manifold and $\{X_i\}_I$ a collection of analytic hypersurfaces in X. $\{X_i\}_I$ is locally finite if for all $p \in X$ exists an open neighbourhood U_p that intersects only a finite number of X_i.

Definition. A Weil divisor is a formal sum $D = \sum a_i X_i$ where $a_i \in \mathbb{Z}$ and $\{X_i\}_I$ is locally finite. We denote with $Div^W(X)$ the free abelian group of Weil divisors on X.

We want now to relate the notion of divisors with the meromorphic functions. Given $f \in H^0(\mathcal{M}^*)$ we can build $div(f)$ as the formal sum of the hypersurfaces representing the zero locus and the pole locus, each one counted with its multiplicity. More precisely:
STEP 1: If f is holomorphic and $\bar{X} = \{g = 0\}$ is an analytical hypersurface defined as the zero locus of an irreducible holomorphic function g, we define for all $p \in \bar{X}$

$$ord_X(f)(p) := \max\{n \geq 0 \mid f_{|U_p} = g^n h, h \in \mathcal{O}^*(U_p)\}$$

Nullstellensatz theorem ensure that $n < \infty$ and that $n > 0$ if and only if $\bar{X} \subseteq Z(f)$. Moreover one can show that the function $ord_X(f)(\cdot)$ is constant on the set of smooth point of $\bar{X}$ and hence, since smooth points are dense and connected in $\bar{X}$, constant on $\bar{X}$. With a great fantasy, we call $ord_X(f)(\bar{X})$ the order of f in $\bar{X}$.
STEP 2:: If f is meromorphic, locally, $f = f_1/f_2$ with f_i holomorphic. We define $ord_X(f)(p) = ord_X(f_1)(p) - ord_X(f_2)(p)$.
STEP 3:

$$div(f) = \sum ord_X(f)(\bar{X}_i)\bar{X}_i.$$

6.2 Cartier Divisors

Cartier divisors are more abstract, but they have the advantage to be described via sheaves. In a first step we will see the definition without using sheaves, and then we introduce them. Both definition will be useful.

Definition. A family (U_α, h_α), where (U_α) is an open cover of X and $h_\alpha \in H^0(U_\alpha, \mathcal{M}^*)$, is admissible if $h_\alpha/h_\beta \in H^0(U_\alpha \cap U_\beta, \mathcal{O}^*)$ for all α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. Two families are equivalent if their union is still admissible.

Definition. A Cartier divisor is a class of equivalent admissible families. We denote by $Div^C(X)$ the set of Cartier divisors.

We would like to put a group operation on $Div^C(X)$: first of all notice that if $D = (U_\alpha, h_\alpha)$ is a divisor, then the family $(U_\alpha, 1/h_\alpha)$ is admissible and define a divisor that we denote by $-D$. Moreover if $D' = (U'_\alpha, h'_\alpha)$ is an other divisor, the family $(U_\alpha \cap U'_\beta, h_\alpha h'_\beta)$ is an admissible family which defines the divisor $D + D'$.

Definition. We say that a divisor D is effective if it admits a representation (U_α, h_α) in which $h_\alpha \in H^0(U_\alpha, \mathscr{O})$ for all α.

Notice that the definition is well posed because every other equivalent family satisfies this property.

Definition. We say that a divisor D is principal if it admits the representation (X, h) in which $h \in H^0(X, \mathcal{M}^*)$, i.e. $D = div(h)$.

Proposition 6.1. *Every divisor is the difference of two effective divisors.*

Sketch of proof. If $D = (U_\alpha, h_\alpha)$, for all $x \in U_\alpha$, there exists a neighbourhood $U_{\alpha,x}$ such that $h_{\alpha|U_{\alpha,x}} = f_{\alpha,x}/g_{\alpha,x}$ with $f_{\alpha,x}, g_{\alpha,x}$ holomorphic. It is sufficient to prove that the families $(U_{\alpha,x}, f_{\alpha,x})$ and $(U_{\alpha,x}, g_{\alpha,x})$ are admissible and that D is the difference of the divisors induced by these families. □

Before using sheaves to describe Cartier divisors, recall that there exists an exact sequence on sheaves over a complex manifold X:

$$0 \longrightarrow \mathscr{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathscr{O}^* \longrightarrow 0$$

Remark. There is a different use of the ash: the sheaf $\mathscr{O}^*$ is the sheaf of holomorphic and nowhere vanishing function. The sheaf $\mathcal{M}^*$ is the sheaf of meromorphic function non identically zero.

From the definition given before, we have that if (U_α, h_α) is admissible

- $h_\alpha \in H^0(U_\alpha, \mathcal{M}^*)$
- $h_\alpha/h_\beta \in H^0(U_\alpha \cap U_\beta, \mathscr{O}^*)$.

Hence $\{h_\alpha\} \in C^0(X, \mathcal{M}^*)$, the group of 0-Čech-cochain, and $\delta\{h_\alpha\} = \{h_\alpha/h_\beta\} \in C^1(X, \mathscr{O}^*)$. If we then consider the quotient sheaf $\mathcal{M}^*/\mathscr{O}^*$ and denote with $\overline{\{h_\alpha\}}$ the equivalent class, we have that a family (U_α, h_α) is admissible if and only if $\overline{\{h_\alpha\}} \in C^0(X, \mathcal{M}^*/\mathscr{O}^*)$ is closed, i.e. $\delta\overline{\{h_\alpha\}} = \overline{0} \in C^1(X, \mathcal{M}^*/\mathscr{O}^*)$, that is $\{h_\alpha/h_\beta\} \in C^1(X, \mathscr{O}^*)$. In conclusion we have that

$$Div^C(X) \cong H^0(X, \mathcal{M}^*/\mathscr{O}^*).$$

Let us denote with $[\overline{\{h_\alpha\}}]$ an element in $H^0(X, \mathcal{M}^*/\mathscr{O}^*)$. (sorry for the horrible notation)

Remark. Given two admissible families (U_α, h_α) and (U'_α, h'_α), we have that

$$[\overline{\{h_\alpha\}}] = [\overline{\{h'_\alpha\}}] \iff [\overline{\{h_\alpha\}} - \overline{\{h'_\alpha\}}] = [\overline{0}] \iff \overline{\{h_\alpha\}} - \overline{\{h'_\alpha\}} = \overline{0} \iff \{h_\alpha/h'_\alpha\} \in C^0(X, \mathscr{O}^*),$$

thus the families are equivalent.

Remark. To be more precise, all the implications in the above remark hold up to take a common refinement of (U_α) and (U'_α), but it always exists since we are working on complex manifolds.

Let us recall that the exact sequence of sheaves above induces an exact sequence in cohomology:

$$0 \longrightarrow H^0(X, \mathcal{O}^*) \longrightarrow H^0(X, \mathcal{M}^*) \xrightarrow{\varphi} H^0(X, \mathcal{M}^*/\mathcal{O}^*) \longrightarrow H^1(X, \mathcal{O}^*) \longrightarrow \dots$$

Definition. A divisor $D \in Im(\varphi)$ is a principal divisor.

6.3 Link between Weil and Cartier

In this subsection we would like to establish an isomorphism between Cartier and Weil divisors. To do so we need the notion of support of a Cartier divisor.

Definition. Let $D \in Div^C(X)$ an effective divisor corresponding to the family (U_α, h_α). Let $F_\alpha \subseteq U_\alpha$ be the closed set defined by $h_\alpha = 0$. It holds that

- $F_\alpha \cap U_\beta = F_\beta \cap U_\alpha$
- $\bigcup_\alpha F_\alpha$ is closed in X
- $\bigcup_\alpha F_\alpha$ is independent from the representation chosen for D.

We define the support of D as $supp(D) := \bigcup_\alpha F_\alpha$.

Theorem 6.2. *Let X be a complex connected compact manifold. There exists an isomorphism*

$$F\colon H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to Div^W(X).$$

Sketch of proof. For the proposition 6.1 every Cartier divisor is $D = D_1 - D_2$ with D_i effective. Let $F(D) = supp(D_1) - supp(D_2)$ counted with multiplicity as shown previously.
This map is clearly a homomorphism, to show that it is bijective we build the inverse: let us consider $D = \sum a_i X_i = D_1 - D_2 = \sum b_j X_j - \sum c_j X_j$ with $b_j, c_j \geq 0$. We set $F^{-1}(D_1) := (U_i, h_i)$ where $X_i \subseteq U_i$ and $h_i = (f_i)^{n_i}$ with f_i is the (unique by Nullstellensatz) function such that $X_i = \{f_i = 0\}$. The thesis follows by setting $F^{-1}(D) = F^{-1}(D_1) - F^{-1}(D_2)$. □

7 Meromorphic Functions

7.1 Theta Functions

Definition. Let V be a vector space and $\Gamma \subseteq V$ a lattice. A theta-function associated with Γ is a non null integer function θ such that, for all $\gamma \in \Gamma$, exists

- a linear form a_γ
- a constant b_γ

such that

$$\theta(z+\gamma) = e^{2\pi i\left(a_\gamma(z)+b_\gamma\right)}\theta(z) \quad \text{for all } z \in V$$

Definition. Given a theta function, we call $(a_\gamma, b_\gamma)_\gamma$ its type.

The main purpose of this subsection is to find a better equivalent representation for the type of a theta function. In the end of the section we will prove that every meromorphic function on a complex torus is the quotient of two theta functions of the same type.

Remark. The constant b_γ is defined up to integer.

Proposition 7.1. *The following properties hold:*

(1) $a_{\gamma_1+\gamma_2} = a_{\gamma_1} + a_{\gamma_2}$

(2) $b_{\gamma_1+\gamma_2} = b_{\gamma_1} + b_{\gamma_2} + a_{\gamma_2}(\gamma_1) \ mod\ \mathbb{Z}$.

Thanks to (1) we can define a application that is $\mathbb{Z}$-linear on first component and $\mathbb{C}$-linear on the second:

$$\begin{aligned} a\colon \Gamma \times V &\to \mathbb{C} \\ (\gamma, z) &\mapsto a_\gamma(z). \end{aligned}$$

We can extend it to an application $a\colon V \times V \to \mathbb{C}$ that is $\mathbb{R}$-linear on first component and $\mathbb{C}$-linear on the second.

Proposition 7.2. *The alternating $\mathbb{R}$-bilinear form $\omega(z,w) := a(z,w) - a(w,z)$ is real, integer on Γ and is an element of $\Lambda^{1,1}V^*_{\mathbb{C}}$ (i.e. $\omega(iz,iw) = \omega(z,w)$).*

Proof. Let $\gamma_1, \gamma_2 \in \Gamma$.

$$\omega(\gamma_1,\gamma_2) = a(\gamma_1,\gamma_2) - a(\gamma_2,\gamma_1) = a_{\gamma_1}(\gamma_2) - a_{\gamma_2}(\gamma_1) \in \mathbb{Z}$$

hence ω is integer on Γ and thus real. Moreover

$$\begin{aligned} \mathbb{R} \ni \omega(iz,iw) - \omega(z,w) =& a(iz,iw) - a(iw,iz) - a(z,w) + a(w,z) = \\ &= ia(iz,w) - ia(iw,z) + ia(z,iw) - ia(w,iz) = \\ &= i(\omega(iz,w) + \omega(z,iw)) \in \mathbb{C}\setminus\mathbb{R}, \end{aligned}$$

and hence $\omega(iz,iw) = \omega(z,w)$ that means that ω is of type $(1,1)$. □

Definition. The form ω is the Riemann form of the theta function θ.

Remark. We emphasize the fact that every Riemann form of a theta function is an integer form. It will be important in the following chapters.

Definition. We say that two theta functions θ_1 and θ_2 are equivalent if and only if they have the same Riemann form.

Definition. A trivial theta function is of the form $\theta(z) = e^{Q(z)}$ where Q is a polynomial of degree at most 2.

Proposition 7.3. *Let θ be a theta function. It is trivial if and only if one of these equivalent conditions hold.*

(1) Its Riemann form is null.

(2) It never vanishes.

Proof. (1) If θ is trivial,

$$\theta(z) = e^{Q(z)} = e^{2\pi i(q(z)+l(z)+c)}$$

with q quadratic and l linear. Hence

$$\begin{aligned}\theta(z+\gamma) =& e^{2\pi i(q(z+\gamma)+l(z)+l(\gamma)+c)} = \\ =& e^{2\pi i(2B(z,\gamma)+q(z)+q(\gamma)+l(z)+l(\gamma)+c)} = \\ =& e^{2\pi i(2B(z,\gamma)+q(\gamma)+l(\gamma))}\theta(z)\end{aligned}$$

where B is the symmetric form associated with q. This implies that $0 = \frac{1}{2}\omega(z,w) := B(z,w) - B(w,z)$.

(2) If θ is trivial it is clear it never vanishes. The converse also holds because we can consider the exact sequence $0 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0$, and hence if $\theta \in H^0(\mathscr{O}^*)$ it means that $\theta(z) = e^{g(z)}$ for some holomorphic g. Moreover, since θ is a theta function,

$$e^{g(z+\gamma)} = e^{a_\gamma(z)+b_\gamma}e^{g(z)}$$

and hence

$$g(z+\gamma) - g(z) = a_\gamma(z) + b_\gamma \implies \frac{d^2}{dx^2}(g(z+\gamma) - g(z)) = 0.$$

Thus $\frac{d^2}{dx^2}g$ is periodic, hence constant, hence g is quadratic. □

Recall that, as we saw in the lemma 3.3, there is an Hermitian form $H(z,w) := \omega(z,iw) - i\omega(z,w)$ associated to ω.

Definition. A normalized theta function is a theta function such that

- $a = \frac{1}{2i}H$
- $Imb_\gamma = -\frac{1}{4}H(\gamma,\gamma)$.

Proposition 7.4. *Every theta function is equivalent to a normalized theta function and a normalized theta function satisfies*

$$\theta(z+\gamma) = e^{2\pi i(\frac{1}{2i}H(\gamma,z)+Reb_\gamma-\frac{1}{4}H(\gamma,\gamma))}\theta(z) = e^{2\pi iReb_\gamma}e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\theta(z) = \alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\theta(z)$$

Remark. We can see $\alpha(\gamma) = e^{2\pi i Reb_\gamma}$ as a function $\alpha\colon \Gamma \to U(1)$. It will assume an important role in the definition of the group $Pic^0(X)$.

Definition. Let θ be a theta function. The couple (H, α) is the type of θ. We call α a semicharacter for H.

Proposition 7.5. *Let α be a semicharacter for H. It holds that, for $\lambda, \mu \in \Gamma$*

$$\alpha(\lambda + \mu) = \alpha(\lambda)\alpha(\mu)e^{\pi i\omega(\lambda,\mu)} = \alpha(\lambda)\alpha(\mu)(-1)^{\omega(\lambda,\mu)}$$

Remark. The type of a theta function is uniquely determined by the quantity

$$\frac{\theta(z+\gamma)}{\theta(z)} = \alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)} := e_\gamma(z)$$

which will turn out to be very important to talk about line bundles.

Definition. The function e_γ is called multiplier for a theta function of type (H, α). In literature it is also called factor of automorphy.

Proposition 7.6. *For all theta functions, the form ω is positive, in the sense that $\omega(x, ix) \geq 0$ for all $x \neq 0$.*

Proof. We can suppose that the theta function is normalized. We set

$$\varphi(z) = e^{-\frac{\pi}{2}H(z,z)}\theta(z)$$

So, for all $\gamma \in \Gamma$,

$$\varphi(z+\gamma) = e^{i\pi(\omega(\gamma,z)+2Reb_\gamma)}\varphi(z).$$

Hence the function $|\varphi|$ is Γ-periodic and hence bounded: there exists a constant K such that $|\theta(z)| \leq Ke^{\frac{\pi}{2}H(z,z)}$. So, if $H(z_0, z_0) < 0$, the holomorphic function $t \mapsto \theta(tz_0)$ goes to 0 when $|t|$ goes to infinity and hence it is identically zero for Liouville theorem. Since the condition $H(z_0, z_0) < 0$ implies $H(z, z) < 0$ in a neighbourhood of z_0, $\theta = 0$ and it is a contradiction. □

We said that two theta functions are equivalent if and only if they have the same Riemann form. Let N be the kernel of H (we set by definition N as the kernel of ω). Let $z_0 \in V$ and θ a normalized theta function, for all $z \in N$ we have:

$$|\theta(z_0 + z)| \leq Ke^{\frac{\pi}{2}H(z_0,z_0)}.$$

Hence the function $z \mapsto \theta(z_0 + z)$ is constant on N. In particular this is an another way to say that a theta function with null Riemann form is trivial.

Remark. If the form ω is such that $\omega(x, ix) > 0$, it is an integer Kähler form on X.

Let us conclude with a proposition that will turn out to be very important in the study of those complex tori which do not admit an integer Kähler form.

Proposition 7.7. *Let θ be a theta function over a complex vector space V associated to a lattice Γ. Let us denote by ω its Riemann form and $N := \ker\omega$. The image of Γ in $V_N := V/N$ is a lattice Γ_N, and θ comes from a non degenerate theta function on V_N associated to the lattice Γ_N.*

Proof. Let $(\gamma_1, \dots, \gamma_{2g})$ be a basis for Γ. There exists by continuity a neighbourhood U of the origin in V_N such that $|H(z, \gamma_j)| < 1$ for all j and for all $z \in V$ whose projection on V_N lies in U. If the projection of z is in $\Gamma_N \cap U$, then $\omega(z, \gamma_j)$ is integer and and its absolute value is strictly less than one and hence null. It implies $z \in N$ and thus $\Gamma_N \cap N = \{0\}$ i.e. Γ_N is discrete in V_N. The vector space spanned by Γ_N is the projection of the vector space spanned by Γ and hence is the whole V_N. This implies that Γ_N is a lattice. By the definition of V_N it is clear that, if $p \colon V \to V_N$, the function $\theta \circ p^{-1}$ is a well defined and non degenerate theta function on V_N associated with Γ_N. □

7.2 Fundamental Theorems and Consequences

Theorem 7.8. *Every effective divisor on a complex torus is a divisor associated with a theta function.*

Proof. Let $D = (h_\alpha, U_\alpha)$ an effective divisor on the complex torus $X = V/\Gamma$. We can suppose that every connected component of $\pi^{-1}(U_\alpha)$ is convex and disjoint by its transaltion by $\Gamma \setminus \{0\}$. Since the functions h_α are holomorphic, the differential forms $\omega_{\alpha\beta} := d\log(h_\alpha/h_\beta)$ are closed and of type (1,0) on $U_\alpha \cap U_\beta$. If we consider (φ_α) a partition of unity relative to (U_α), the form $\omega_\alpha := \sum_\delta \varphi_\delta \omega_{\alpha\delta}$ is of type (1,0) on U_α. On $U_\alpha \cap U_\beta$ we have

$$\omega_\alpha - \omega_\beta = \omega_{\alpha\beta} \;\text{ and }\; d\omega_\alpha = d\omega_\beta.$$

Thus $(d\omega_\alpha)$ define a unique closed 2-form without terms of type (0,2) that is in the same cohomology class of a constant form η of the same type. It means that does exist a global 1-form ω_0 of type (1,0) such that on each U_α

$$d\omega_\alpha = \eta + d\omega_0.$$

Since $\pi^{-1}(U_\alpha)$ is simply connected we have that $\pi^*\eta = d\omega_1$ for some $\omega_1 = \sum_j l_j(z)dz_j$ and hence, since $d\Big(\pi^*(\omega_\alpha - \omega_0) - \omega_1\Big) = 0$, there exist a function f_α such that

$$\pi^*(\omega_\alpha - \omega_0) - \omega_1 = df_\alpha.$$

Since all terms in left side are of type (1,0), the functions f_α are holomorphic. We now would like to find a special global function starting from this local data. On $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$,

$$df_\alpha - df_\beta = d\log \pi^*(h_\alpha/h_\beta),$$

hence, up to multiplication for a constant in any connected component, the functions $e^{-f_\alpha}\pi^* h_\alpha$ glue together to a holomorphic function θ on V. We have now to prove that θ is actually a theta function. On $\pi^{-1}(U_\alpha)$ we have that

$$\log \frac{\theta(z+\gamma)}{\theta(z)} = f_\alpha(z) - f_\alpha(z+\gamma),$$

so, since $\pi^*(\omega_\alpha - \omega_0)$ is Γ periodic,

$$d\log \frac{\theta(z+\gamma)}{\theta(z)} = \omega_1(z+\gamma) - \omega_1(z) = \sum_j l_j(\gamma)dz_j$$

that shows that θ is a theta function with $a_\gamma = 1/2\pi i \sum_j l_j(\gamma)z_j$. □

Definition. We set the type of a effective divisor as the type of its associated theta function.

Theorem 7.9. *Every non null meromorphic function on a complex torus is a quotient of two theta functions of the same type.*

Proof. Let f be a non null meromorphic function over X. We can decompose its divisor as a difference of two effective divisors $D = D_1 - D_2$ and let θ be a theta function associated with D_2. The function $\theta \cdot (f \circ \pi)$ is holomorphic on V because its divisor D_1 is effective and it is a theta function of the same type of θ. $\square$

We end this part giving three results that bind Kähler forms, meromorphic functions, divisors and projectivity of a complex tori.

Proposition 7.10. *Let X be a complex torus and $u\colon X \longrightarrow \mathbb{CP}^n$ a holomorphic function. There exist $\theta_0, \ldots, \theta_n$ holomorphic such that one of the following mutually exclusive condition holds*

- *they are null.*
- *they are normalized theta functions of the same type, without common zeroes and such that $u(x) = (\theta_0(x), \ldots, \theta_n(x))$ for all $x \in X$.*

Proof. Let us suppose the image of u does not lie in any hyperplane $x_j = 0$. For all j, the equation $x_j = 0$ defines an effective divisor on $\mathbb{P}^n$ and, in the same way, the equation $x_j \circ u = 0$ defines an effective divisor D_j on X. Let θ_0 be a normalized theta function associated with D_0. For all j, the function $\theta_j := \frac{x_j \circ u}{x_0 \circ u}\theta_0$ is a normalized theta function ($\frac{x_j \circ u}{x_0 \circ u}$ is Γ-periodic) of the same type as θ_0 (it will be clear later, see for example section 8.5) and associated with the effective divisor D_j. This implies that $u = (x_0 \circ u, \ldots, x_n \circ u) = (\theta_0, \ldots, \theta_n)$. $\square$

Proposition 7.11. *Let X be a complex torus and let us suppose there exists a holomorphic function $u\colon X \longrightarrow \mathbb{CP}^n$ and a point $x \in X$ such that $u^{-1}(u(x))$ is finite. So there exists an integer Kähler form on X.*

Proof. Thanks to the previous proposition there exist $\theta_0, \ldots, \theta_n$, normalized non null theta functions of the same type and with no common zeroes such that $u = (\theta_0, \ldots, \theta_n)$. Let ω be their common Riemann form and N its kernel. Since fibres of u are finite, N must be empty because $\theta_j(z_0 + z) = a_j$ is constant for all $z \in N$ and it would imply that $u^{-1}(a_0, \ldots, a_n)$ is not finite. Hence ω is an integer Kähler form. $\square$

Proposition 7.12. *Let X be a complex torus. There exists a complex torus X_{ab} (called the abelianization of X) and a holomorphic surjection $\rho\colon X \longrightarrow X_{ab}$ such that:*

- *There exists an integer Kähler form on X_{ab}.*
- *Every holomorphic function from X to a projective space factorises by ρ.*
- *ρ induces an isomorphism between the fields of meromorphic functions $\mathcal{M}(X_{ab})$ and $\mathcal{M}(X)$.*
- *ρ induces an isomorphism between the groups $Div(X_{ab})$ and $Div(X)$.*

We will see the proof of this fact in section 10.

8 Line Bundles

8.1 Standard Definition

For the convenience of the reader we recall basic definitions and results about line bundles over a connected complex manifold. The main goal is to give a description of this theory via sheaves in order to make clear its bond with divisors and meromorphic functions.

Definition. A line bundle over X is the pair (L, p) where L is a connected complex manifold and $p\colon L \to M$ is a holomorphic surjective function such that exists a cover $\{U_\alpha\}$ of X and a family of isomorphisms $\psi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ with transition maps

$$\begin{aligned} \psi_\alpha \circ \psi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{C} &\to (U_\alpha \cap U_\beta) \times \mathbb{C} \\ (z, t) &\longmapsto (z, g_{\alpha\beta}(z)t) \end{aligned}$$

where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}$ are holomorphic non-vanishing functions, i.e. $g_{\alpha\beta}(z) \in GL_1(\mathbb{C}) = \mathbb{C}^*$ for all $z \in U_\alpha \cap U_\beta$.

Remark. The functions $g_{\alpha\beta}$ define completely the line bundle and satisfy cocycle conditions, in fact, if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$:

$$\begin{gathered} id = \psi_\alpha \circ \psi_\alpha^{-1} = \psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ \psi_\gamma^{-1} \circ \psi_\gamma \circ \psi_\alpha^{-1} \\ (z,t) \mapsto (z, g_{\alpha\alpha}(z)t) = (z, g_{\alpha\beta}(z) g_{\beta\gamma}(z) g_{\gamma\alpha}(z)t) = (z, t) \end{gathered}$$

Definition. A global section of a line bundle is a holomorphic function $s\colon X \to L$ such that $p \circ s = id$.

Notice that we can use the trivialising atlas to express a section s as a collection of local data: $s = \{s_\alpha\colon U_\alpha \to U_\alpha \times \mathbb{C}\}$. To be more precise:

$$\begin{aligned} s_\alpha = (id, f_\alpha)\colon U_\alpha &\to U_\alpha \times \mathbb{C} \\ z &\mapsto (z, f_\alpha(z)). \end{aligned}$$

Remark. If $U_\alpha \cap U_\beta \neq \emptyset$,

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times \mathbb{C} \arrow[r, "\psi_\alpha \circ \psi_\beta^{-1}"] & (U_\alpha \cap U_\beta) \times \mathbb{C} \\ U_\alpha \cap U_\beta \arrow[u, "s_\alpha"] \arrow[r, "id"] & U_\alpha \cap U_\beta \arrow[u, "s_\beta"'] \end{tikzcd} \qquad \begin{tikzcd} f_\alpha(z) \arrow[r, mapsto, "pr_2 \circ \psi_\alpha \circ \psi_\beta^{-1}"] & g_{\alpha\beta}(z) f_\alpha(z) = f_\beta(z) \\ z \arrow[u, mapsto] \arrow[r, mapsto] & z \arrow[u, mapsto] \end{tikzcd}$$

Therefore the transformation law for local sections coming form the same global section is $f_\beta(z) = g_{\alpha\beta}(z) f_\alpha(z)$.

Definition. Two line bundles $p\colon L \to X$ and $p'\colon L' \to X$ are isomorphic if there exists an isomorphism $u\colon L \to L'$ such that $p \circ u = p'$ and that is linear on the fibres.

Remark. We would like to give a local description of a line bundle isomorphism: first of all notice that we can suppose, up to take a common refinement, that L and L' are trivialized via the same open cover that means that both L and L' are modelled on $U_\alpha \times \mathbb{C}$.

$$\begin{tikzcd}
L \arrow[r, "u"] \arrow[d, "\psi_\alpha"] & L' \arrow[d, "\psi'_\alpha"] \\
U_\alpha \times \mathbb{C} \arrow[r, "(id{,}h_\alpha)"] & U_\alpha \times \mathbb{C}
\end{tikzcd}$$

Thus, $u(\psi_\alpha^{-1}(z,t)) = \psi_\alpha'^{-1}(z, h_\alpha(x)t)$, where $h_\alpha \colon \mathbb{C} \to \mathbb{C}$ is a linear map.
Hence we want that locally the following diagram commutes for all $U_\alpha \cap U_\beta \neq \emptyset$.

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times \mathbb{C} \arrow[r, "(id{,}h_\beta)"] \arrow[d, "(id{,}g_{\alpha\beta})"] & (U_\alpha \cap U_\beta) \times \mathbb{C} \arrow[d, "(id{,}g'_{\alpha\beta})"] \\
(U_\alpha \cap U_\beta) \times \mathbb{C} \arrow[r, "(id{,}h_\alpha)"] & (U_\alpha \cap U_\beta) \times \mathbb{C}
\end{tikzcd}$$

And it means that $g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}$.

8.2 Sheaf-theoretic definition

We would like to transpose these notions in therms of sheaf cohomology.

Definition. A holomorphic line bundle is a closed 1-cochain $\{g_{\alpha\beta}\} \in C^1(X, \mathscr{O}^*)$.

Indeed if we compute $\delta\{g_{\alpha\beta}\} = g_{\alpha\beta}(g_{\gamma\beta})^{-1} g_{\gamma\alpha} = g_{\alpha\alpha} = 0$, we get the cocycle conditions and hence $\{g_{\alpha\beta}\}$ defines uniquely a line bundles in the standard sense.

Definition. Two line bundles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are isomorphic if and only if they belong to the same cohomology class in $H^1(X, \mathscr{O}^*)$.

Proposition 8.1. *The two notions of isomorphism coincide.*

Proof. Let us denote with $[\cdot]$ the homology class.

$$[\{g'_{\alpha\beta}\}] = [\{g_{\alpha\beta}\}] \iff \{g'_{\alpha\beta}\} = \{g_{\alpha\beta}\} \cdot \delta\{h_\alpha\} = \{h_\alpha g_{\alpha\beta} h_\beta^{-1}\}$$

with $\{h_\alpha\} \in C^0(X, \mathscr{O}^*)$. □

Hence, up to isomorphism, holomorphic line bundle are classified by $H^1(X, \mathscr{O}^*)$.

Remark. Recall that we defined a section s as a collection $\{s_\alpha = (id, f_\alpha) \colon U_\alpha \to U_\alpha \times \mathbb{C}\}$ with $f_\alpha \in \mathscr{O}(U_\alpha)$ such that $f_\alpha = g_{\alpha\beta} \cdot f_\beta$. This is possible if and only if f_α and f_β vanish in the same set in $U_\alpha \cap U_\beta$ since $g_{\alpha\beta} \in \mathscr{O}^*(U_\alpha \cap U_\beta)$ and $g_{\alpha\beta} = f_\alpha / f_\beta$. Hence, as cochains, $\{g_{\alpha\beta}\} = \delta\{f_\alpha\}$ that is clearly closed but not necessarily exact since $\{f_\alpha\} \in C^0(X, \mathscr{O})$ and, a priori, not in $C^0(X, \mathscr{O}^*)$.

We want to give $H^1(X, \mathscr{O}^*)$ a group structure:

Definition. Given two line bundles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ we define:

- The dual line bundle L^{-1} associated with $\left\{\frac{1}{g_{\alpha\beta}}\right\}$.
- The tensor product line bundle $L_1 \otimes L_2$ associated with $\{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$.

Who is the identity element? It is clear that the trivial bundle with identical transition maps $\{1_{\alpha\beta}\}$ acts as identical element, but, by the definition of isomorphic line bundles, any other bundle in its cohomology class is a possible representative. Hence

$$[\{1_{\alpha\beta}\}] = B^1(X, \mathcal{O}^*) = \left\{\{g_{\alpha\beta}\} = \{h_\alpha/h_\beta\} \text{ with } \{h_\alpha\} \in C^0(X, \mathcal{O}^*)\right\}$$

is the identity element in $H^1(X, \mathcal{O}^*)$.

Definition. We denote by $Pic(X)$ the group $H^1(X, \mathcal{O}^*)$ endowed with the operations just defined. And we call it the Picard group of X.

The Picard group is a very strong tool to study complex manifolds because line bundles encode a lot of information of the underlying complex structure. The main reason of their importance, as we will see, is related to the fact that on a compact complex manifold there are no but constant global holomorphic function that can not give us any information. We can so use, when they exist, global sections of line bundles (that we can locally see in some ways as holomorphic maps $X \to \mathbb{C}$) to pull out the informations we need. For example in the next section we will see how line bundles are strictly related to divisors and in section 10 we will relate the projectivity of a torus (but in general of a compact Kähler manifold) to the existence of an ample line bundle.

8.3 Divisors and Line Bundles

The language of sheaves helps us to find a very strong link between divisors and line bundles.

Definition. Let $D = (U_\alpha, h_\alpha)$ be a divisor. We denote by $\mathcal{O}_X(D)$ the line bundle with transition maps $g_{\alpha\beta} = h_\alpha/h_\beta$.

We can define a group morphism

$$\begin{aligned} \Phi\colon H^0(X, \mathcal{M}^*/\mathcal{O}^*) = \ Div(X) &\to H^1(X, \mathcal{O}^*) = Pic(X) \\ D &\longmapsto \mathcal{O}_X(D). \end{aligned}$$

It is well defined because, if (U_α, h'_α) is an another family that describes D (i.e. equivalent to (U_α, h_α)), we have shown that $h_\alpha/h'_\alpha = l_\alpha$ is holomorphic nowhere vanishing. Thus, if $g'_{\alpha\beta} = h'_\alpha/h'_\beta$, we have that $\{g'_{\alpha\beta}\} = \{g_{\alpha\beta}\}\delta\{l_\alpha\}$ and hence they belong to the same cohomology class.

Lemma 8.2. *Principal divisors form the kernel of* Φ.

Proof. Recall that a principal divisor is $D = div(h)$ with $h \in H^0(X, \mathcal{M}^*)$. Hence if we consider the cochain $\{h_\alpha/h_\beta\}$ it turns out to be in $C^1(X, \mathcal{O}^*)$ since $h_\alpha/h_\beta = id$. The conclusion follows from the definition of the identity element in $Pic(X)$. □

Proposition 8.3. *The morphism* $\Phi\colon D \mapsto \mathcal{O}_X(D)$ *establishes an isomorphism between divisors modulo principal divisors and line bundles (modulo equivalence) which admit a non trivial meromorphic section.*

Indeed, a meromorphic section of a line bundle is the data $\{s_\alpha \in \mathcal{M}^*(U_\alpha)\}$ such that $s_\beta = g_{\alpha\beta}s_\alpha$ and $\{g_{\alpha\beta}\} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. Every divisor $D = (U_\alpha, h_\alpha)$ allows us to associate $\mathcal{O}_X(D)$ a meromorphic section that is $\{h_\alpha\}$. It is in fact a section and defines uniquely the line bundle by the definition, since $g_{\alpha\beta} = h_\alpha/h_\beta$.

Proposition 8.4. *For a divisor D on X we have that*

$$\Gamma(X, \mathcal{O}_X(D)) \cong \{f \in \mathcal{M}(X) \mid f = 0 \text{ or } div(f) + D \geq 0\}$$

Indeed, if (U_α, h_α) is a representation for D and f is a meromorphic function such that $div(f)+D$ is effective, the function fh_α is holomorphic on U_α and then it defines a section of $\mathcal{O}_X(D)$. On the other hand, to every section (s_α) of $\mathcal{O}_X(D)$, we can associate the meromorphic function f locally defined by s_α/h_α.

Let L be a line bundle. We denote by $|L|$ the set of effective divisors associated with non null holomorphic sections of L. In fact we recall that a holomorphic section for L is a collection $(s_\alpha \in \mathcal{O}(U_\alpha))$ such that $s_\alpha = g_{\alpha\beta}s_\beta$ and $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. On the other hand an effective divisor is a divisor $D = (h_\alpha, U_\alpha)$ where $h_\alpha \in \mathcal{O}(U_\alpha)$ and $h_\alpha/h_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. By the fact that we consider a divisor D up to equivalent admissible representation, we can consider the map

$$\mathbb{P}\Gamma(X, L) \to Div(X)$$

since (s_α) and (λs_α) would generate the same divisor. Since a torus is compact the map $div\colon \mathbb{P}\Gamma(X, L) \to |L|$ is bijective.

8.4 Line Bundles on Complex Tori

The most natural way to build line bundles on complex tori is to start from the trivial bundle over the vector space V, that is $V \times \mathbb{C} \to V$ and acts on it via translation by elements of Γ.

$$\begin{aligned}(V \times \mathbb{C}) \times \Gamma &\to V \times \mathbb{C}\\ (z, t, \gamma) &\longmapsto (z + \gamma, e_\gamma(z) \cdot t)\end{aligned}$$

where $e_\gamma\colon V \to \mathbb{C}$ is a holomorphic function, called multiplier or factor of automorphy, with suitable conditions. In fact, so that this is an action, the multipliers e_γ must satisfy

$$e_{\gamma_1+\gamma_2}(z) = e_{\gamma_1}(z + \gamma_2)e_{\gamma_2}(z).$$

Recall that, from proposition 7.4, for a normalized theta function of type (H, α) the quantity

$$e_\gamma(z) = \frac{\theta(z+\gamma)}{\theta(z)} = \alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}$$

satisfies this condition and in this case e_γ is said canonical multiplier. We will see in the next section that we can define an equivalent relation in the group of multipliers such that the quotient space can be seen as a particular cohomology group.

Definition. We denote by $L(H, \alpha)$ the line bundle over X associated with the theta function θ of type (H, α) in the sense that

$$L(H, \alpha) = (V \times \mathbb{C})/\Gamma$$

with the action given by the multiplier $e_\gamma(z) = \theta(z + \gamma)/\theta(z)$.

Thus we associated to every theta function a line bundle. We ask if it is possible to establish a bijective correspondence, i.e. if we can associate to every line bundle a theta function (Appell-Humber Theorem). The answer is yes but, to explain why, we have to wait for stronger tools such as

the first Chern class associated with a line bundle.
Sometimes will be useful to use a different notation: if $e_\gamma(z)$ is the multiplier associated with the line bundle $L := L(H,\alpha)$, we will denote $e_\gamma(z) := e_L(\gamma, z) = \alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}$.

Corollary 8.5. *The function e_L has the following properties:*

- $\alpha(n\gamma) = \alpha(\gamma)^n$
- $e_L(\gamma, z + w) = e_L(\gamma, z)e^{\pi H(w,\gamma)}$
- $e_L(\gamma + \mu, z) = e_L(\lambda, \mu + z)e_L(\mu, z)$
- $e_L(\gamma, z)^{-1} = e_L(-\gamma, z)e^{-\pi H(\gamma,\gamma)}$

While waiting for the first Chern class, we can focus our attention to an another important and interesting fact concerning sections in $\Gamma\big(X, L(H,\alpha)\big)$.
Let $\Pi\colon V \times \mathbb{C} \to L(H,\alpha)$ the canonical quotient projection and let s be a section of $L(H,\alpha) \to X$. We can associate to s a holomorphic function $\theta_s\colon V \to \mathbb{C}$ such that the following diagram commutes.

$$\begin{tikzcd}
V \times \mathbb{C} \arrow[rr, "\Pi"] \arrow[dr, "\pi\circ pr_1"'] & & L(H,\alpha) & & (z, \theta_s(z)) \arrow[rr, mapsto, "\Pi"] \arrow[dr, mapsto] & & s(\pi(z)) \\
& X \arrow[ur, "s"'] \arrow[u, phantom, "\circlearrowright"] & & & & \pi(z) \arrow[ur, mapsto] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

Since, by definition of quotient projection, $\Pi(z+\gamma, \theta_s(z+\gamma))$ must be equal to $\Pi(z, \theta_s(z))$, the function θ_s must satisfy

$$\theta_s(z+\gamma) = e_\gamma(z)\theta_s(z) = \alpha(\gamma)e^{\pi H(\gamma,z)+\frac{\pi}{2}H(\gamma,\gamma)}\theta_s(z)$$

and hence it is a theta function.

Proposition 8.6. *The application*

$$\begin{aligned}
\varphi\colon \Gamma\big(X, L(H,\alpha)\big) &\to \{\textit{normalized theta functions of type } (H,\alpha)\} \\
s &\longmapsto \theta_s
\end{aligned}$$

establishes an isomorphism between the vector space of sections of $L(H,\alpha)$ and the set of normalized theta functions of type (H,α).

This is a crucial fact because we have associated to every line bundle $L = L(H,\alpha)$ a family of theta functions of type (H,α). If we prove that every line bundle is of the form $L = L(H,\alpha)$ we will reach our intent.

8.5 Digression: Pull-Back of Line Bundles

Recall that we have the projection $\pi\colon V \to X$ and we can consider the pullback of every line bundle $L \to X$ as a line bundle $\pi^*L \to V$.
There is a general theory concerning line bundles and universal (holomorphic) covering spaces $\pi\colon \tilde{X} \to X$: in particular one can describe line bundles L over X whose pullback π^*L is trivial in terms of the cohomology of the fundamental group $\pi_1(X)$. Let us give some simple definition concerning group cohomology (we will not go through the details).

Definition. A holomorphic map $f: \pi_1(X) \times \tilde{X} \to \mathbb{C}^*$ satisfying the cocycle relation

$$f(\lambda + \mu, z) = f(\lambda, \mu + z) f(\mu, z)$$

is a 1-cocycle of $\pi_1(X)$ with values in $H^0(\tilde{X}, \mathcal{O}^*)$.

Definition. A 1-cocycle of the form $f(\lambda, z) = h(\lambda + z)h(z)^{-1}$ for some $h \in H^0(V, \mathcal{O}^*)$ is a 1-boundary.

Definition. The quotient between 1-cocycles and 1-boundaries is a cohomology group: $H^1\big(\pi_1(X), H^0(\tilde{X}, \mathcal{O}^*)\big)$.

In our case we have:

- $\pi_1(X) = \Gamma$
- $\tilde{X} = V$
- $H^0(\tilde{X}, \mathcal{O}^*) = H^0(V, \mathcal{O}^*) = \mathcal{O}^*(V)$

and moreover any line bundle over V is trivial, hence this classification is complete. Notice that, thanks to the third point of corollary 8.5, the 1-cocycles are exactly the multipliers e_L and then two 1-cocycles that differ by a boundary are associated with the same line bundle because they can be seen as different sections (theta functions) of the same type.

Remark. Any e_L is of the form $\theta(\gamma + z)/\theta(z)$, but, in general, $\theta \notin H^0(V, \mathcal{O}^*)$. According to Proposition 7.3, the only holomorphic nowhere vanishing theta functions are trivial theta function $\theta(z) = e^{Q(z)}$. Thus e_L is a 1-boundary if and only if its theta function is trivial. It makes sense because, for Proposition 7.3, θ is trivial if and only if $\omega = 0$ and it holds if and only if $H = 0$ and hence it is of type $(0, \alpha)$ and it leads to a trivial line bundle in $Pic^0(X)$ as we will see in the next sections.

In literature, there exists a general theorem about line bundles and covering spaces.

Theorem 8.7. *There is a canonical isomorphism*

$$\varphi_1: H^1\big(\pi_1(X), H^0(\tilde{X}, \mathcal{O}^*)\big) \to \ker\big(\pi^*: H^1(X, \mathcal{O}^*) \to H^1(\tilde{X}, \mathcal{O}^*)\big)$$

Proof. See (BL), Proposition B1. □

In the theory of complex tori this theorem becomes much simpler since $H^1(\tilde{X}, \mathcal{O}^*) = Pic(V) = \{0\}$.

Theorem 8.8. *There is a canonical isomorphism*

$$\varphi_1: H^1(\Gamma, H^0(V, \mathcal{O}^*)) \to Pic(X)$$

Remark. This gives us an another way to see Proposition 8.6. In fact for any line bundle L on X there is a natural isomorphism

$$H^0(X, L) \cong H^0(V, \pi^* L)^\Gamma \cong H^0(V, V \times \mathbb{C})^\Gamma$$

where the last equivalence is given by a trivialization $\alpha: \pi^* L \to V \times \mathbb{C}$ that always exists. Hence it follows that if $f \in Z^1\big(\Gamma, H^0(V, \mathcal{O}^*)\big)$ is associated with α, the elements in $H^0(V, V \times \mathbb{C})^\Gamma$ are just the holomorphic functions $\theta: V \to \mathbb{C}$ satisfying

$$\theta(\lambda + z) = f(\lambda, z)\theta(z).$$

After this long digression about pullbacks via the covering map, we would like to study the behaviour of line bundles under pullbacks by holomorphic maps between complex tori. Recall that such holomorphic maps can be seen as composition of homomorphisms and translations.
Now, we must be careful with the notation: here z is an element of V and we denote by $\bar{z}$ the element $\pi(z) \in X$.

Lemma 8.9. *For any $L = L(H, \alpha) \in Pic(X)$ and $\bar{z} \in X$,*

$$\tau_{\bar{z}}^* L(H, \alpha) = L(H, \alpha e^{2\pi i \omega(z, \cdot)})$$

Proof. The translation τ_z on V induces a translation $\tau_{\bar{z}}$ on X. Moreover the induced map $\tau_{\bar{z}*}$ is the identity on $\Gamma = \pi_1(X)$. Hence, if e_L is the multiplier of L, $(id_\Gamma \times \tau_z)^* e_L$ is a multiplier for $\tau_{\bar{z}}^* L(H, \alpha)$. We would like to find a multiplier equivalent to it (i.e. such that they differ for a 1-boundary) that is canonical. For $h(w) = e^{-\pi H(w,z)}$ we have:

$$\begin{aligned}(id_\Gamma \times \tau_z)^* e_L(\lambda, w) h(w + \lambda) h(w)^{-1} &= \\ = e_L(\lambda, w + z) e^{-\pi H(w+\lambda, z) + \pi H(w,z)} &= \\ = \alpha(\lambda) e^{2\pi i \omega(z,\lambda)} e^{\pi H(w,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}\end{aligned}$$

and hence it is equivalent to $(id_\Gamma \times \tau_z)^* e_L$ and canonical. □

Remark. If $L = L(0, \alpha)$, it holds that $\tau_{\bar{z}}^* L(0, \alpha) = L(0, \alpha)$ because $H = \omega = 0$.

Theorem 8.10 (of the Square)**.** *For all $\bar{z}, \bar{w} \in X$ and $L(H, \alpha) \in Pic(X)$,*

$$\tau_{\bar{z}+\bar{w}}^* L(H, \alpha) = \tau_{\bar{z}}^* L(H, \alpha) \otimes \tau_{\bar{w}}^* L(H, \alpha) \otimes L^{-1}(H, \alpha)$$

Proof. It follows from the previous proposition. In fact it holds that $\tau_{\bar{z}+\bar{w}}^* L(H, \alpha) = L(H, \alpha e^{2\pi i \omega(z+w, \cdot)})$ and the canonical multiplier of $\tau_{\bar{z}}^* L(H, \alpha) \otimes \tau_{\bar{w}}^* L(H, \alpha) \otimes L^{-1}(H, \alpha)$ is:

$$\alpha e^{2\pi i \omega(z,\cdot)} \cdot \alpha e^{2\pi i \omega(w,\cdot)} \cdot \alpha^{-1} = \alpha e^{2\pi i \omega(z+w,\cdot)}.$$

□

8.6 First Chern Class and Pic(X)

Le X be a complex manifold and let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{M}^*/\mathcal{O}^* \longrightarrow 0$$

and the induced sequence in cohomology

$$0 \longrightarrow H^0(X, \mathcal{O}^*) \longrightarrow H^0(X, \mathcal{M}^*) \longrightarrow H^0(X, \mathcal{M}^*/\mathcal{O}^*) = Div(X) \xrightarrow{\Phi} H^1(X, \mathcal{O}^*) = Pic(X) \longrightarrow \dots$$

in which appear the groups $Div(X)$ and $Pic(X)$.
It is clear that the image of $H^0(X, \mathcal{M}^*)$, isomorphic to the group of principal divisors, in $H^0(X, \mathcal{M}^*/\mathcal{O}^*)$ is in the kernel of Φ, i.e. defines trivial line bundles.
This sequence shows us that there exist line bundles which do not admit meromorphic sections. Their existence is in fact related to the surjectivity of $Div(X) \to Pic(X)$ that occurs if and only if $Pic(X) \to$

$H^1(X, \mathcal{M}^*)$ is the null map.
We focus our attention on an another sequence in which $Pic(X)$ occurs: it is the long exact sequence induced by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}^* \longrightarrow 0.$$

Definition. The map

$$c_1 \colon H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z})$$

is the first Chern class.

In order to understand the map c_1 let us consider the C^∞ sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathscr{O} \arrow[r] \arrow[d, hook] & \mathscr{O}^* \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & C^\infty \arrow[r] & (C^\infty)^* \arrow[r] & 0
\end{tikzcd}$$

Since the sheaf C^∞ is acyclic (because partitions of unity are allowed), we have that $c_1 \colon H^1(X, (C^\infty)^*) \to H^2(X, \mathbb{Z})$ is an isomorphism. Hence smooth line bundles are characterised by a discrete invariant and the trivial smooth line bundle is such that $c_1(L) = 0$, thus $c_1(L)$ is an obstruction to the existence of a smooth never vanishing section.
In the holomorphic case, since c_1 is no more an isomorphism, a lot of different line bundles are such that $c_1(L) = 0$, so the smooth trivial line bundle can be endowed by many different holomorphic structures that are parametrized by $Pic^0(X) := \ker(c_1) \cong \cdots \cong H^1(X, \mathscr{O})/H^1(X, \mathbb{Z})$.

We are now ready to study the interaction between $Div(X)$ and $Pic(x)$.
From now, we forget the manifold in the scripture of cohomology groups.

Remark. If X is compact

$$H^0(\mathbb{Z}) = \mathbb{Z} \quad H^0(\mathscr{O}) = \mathbb{C} \quad H^0(\mathscr{O}^*) = \mathbb{C}^*$$

And, since the sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$ is exact, we can split the long exact sequence putting a "0" between $H^0(\mathscr{O}^*)$ and $H^1(\mathbb{Z})$.

Hence, for a compact complex manifold we have

$$\begin{tikzcd}
& 0 \arrow[r] & H^1(\mathbb{Z}) \arrow[r] & H^1(\mathscr{O}) \arrow[dr, bend left] & & & \\
0 \arrow[r] & H^0(\mathscr{O}^*) \arrow[r] & H^0(\mathcal{M}^*) \arrow[r] & Div(X) \arrow[r, "\Phi"] \arrow[drr, dotted, bend right, "\eta"'] & Pic(X) \arrow[r] \arrow[dr, bend right, "c_1"] & H^1(\mathcal{M}^*) \arrow[r] & \dots \\
& & & & & H^2(\mathbb{Z}) \arrow[r] & \dots
\end{tikzcd}$$

Theorem 8.11. *For any compact complex manifold*

$$\eta(D) = PD([D])$$

where $[D] \in H_{2n-2}(X, \mathbb{Z})$ is the homology class of the hypersurface identified by D and $PD \colon H_{2n-2}(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the Poincaré dual.

Application to Complex Tori

We would like to study the first Chern class on complex tori. To do so we need to introduce a Hermitian metric on a line bundle and study the cohomology group of the sheaf $\mathcal{O}$ over a complex torus.

Definition. A Hermitian metric over a line bundle $p\colon L \to X$ is a function $||\cdot||\colon L \to [0,\infty)$ such that in local charts (ψ_α, U_α) one have $||\psi_\alpha^{-1}(z,t)|| = h_\alpha(z)|t|$, where h_α is a smooth positive function.

Remark. If X is a complex torus the function

$$||(z,t)|| = e^{-\frac{\pi}{2}H(z,z)}|t|$$

is an Hermitian metric over the line bundle $L(H,\alpha)$.

Proposition 8.12. *Let s be a non vanishing holomorphic section of the line bundle $L(H,\alpha)$. The $(1,1)$-form given by*

$$\frac{1}{2\pi i}\partial\overline{\partial}\log(||s||^2)$$

is independent by the choice of s and represents the cohomology class $c_1(L(H,\alpha))$.

Sketch of proof. Every other non vanishing section s' of L is of the form $s' = fs$, where f is nowhere vanishing and holomorphic.

$$\partial\overline{\partial}\log(||s'||^2) = \partial\overline{\partial}\log(||s||^2 f\overline{f}) = \partial\overline{\partial}\log(||s||^2) + \partial\overline{\partial}\log(f) + \partial\overline{\partial}\log(\overline{f}) = \partial\overline{\partial}\log(||s||^2)$$

since $\partial\log(\overline{f}) = \overline{\partial}\log(f) = 0$. We do not prove it represents $c_1(L(H,\alpha))$. □

We proved in Proposition 8.6 that, given a line bundle $L = L(H,\alpha)$ and a section $s\colon X \to L$, we can associate to it a normalized theta function of type (H,α). In Proposition 7.2 we associated to every theta function its Riemann form ω. The proof of the following Corollary strengthens the relation between sections of L and its first Chern class showing how ω and s are related.

Corollary 8.13. *The form $\omega = ImH$ represents the first Chern class of the line bundle $L(H,\alpha)$.*

Proof. We can use the metric in the above remark to compute $||s(z)||^2 = ||(z,t)||^2 = e^{-\pi H(z,z)}|t|^2$

$$\frac{1}{2\pi i}\partial\bar{\partial}\log(||s||^2) = \frac{1}{2\pi i}\left(\partial\bar{\partial}\log(e^{-\pi H(z,z)})+\partial\bar{\partial}\log(t\bar{t})\right) = \frac{1}{2\pi i}\partial\bar{\partial}\log(e^{-\pi H(z,z)}) = \frac{i}{2}\partial\bar{\partial}H(z,z) = ImH = \omega$$

□

Before starting with the proof of Appel-Humbert theorem we need to make some remarks about sheaf cohomology groups on X.

Remark. For all $r \geq 0$ it holds that

$$H^r(X,\mathbb{Z}) \cong \Lambda^r\Gamma^* \quad \text{and} \quad H^r(X,\mathcal{O}) \cong \Lambda^r\overline{V}^* \cong \Lambda^{0,r}V^*.$$

Moreover recall that there exists a sheaf inclusion $i\colon \mathbb{Z} \to \mathcal{O}$ given by the composition $\mathbb{Z} \hookrightarrow \mathbb{C} \hookrightarrow \mathcal{O}$. This morphism passes in cohomology $H^r(i)\colon H^r(X,\mathbb{Z}) \to H^r(X,\mathcal{O})$ and associates to an alternating form ω over Γ its $\mathbb{R}$-linear extension $\omega_\mathbb{R}$ over V and then his $(0,r)$ part: $\omega^{0,r}$. Then we have the following:

$$\begin{tikzcd}
0 \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^1(X,\mathscr{O}) \arrow[r] \arrow[d, "\cong"] & H^1(X,\mathscr{O}^*) \arrow[r] \arrow[d, "\cong"] & H^2(X,\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H^2(X,\mathscr{O}) \arrow[d, "\cong"] \\
0 \arrow[r] & \Gamma^* \arrow[r, "H^1(i)"] & \overline{V}^* \arrow[r, "H^1(e)"] & Pic(X) \arrow[r, "c_1"] & \Lambda^2\Gamma^* \arrow[r, "H^2(i)"] & \Lambda^{0,2}V^*
\end{tikzcd}$$

Where the functions are explicitly defined by

- $H^1(i)\colon \omega \mapsto \omega_{\mathbb{R}}^{0,1}$
- $H^1(e)\colon l \mapsto L(0, e^{2\pi i Iml(\cdot)})$
- $H^2(i)\colon \omega \mapsto \omega_{\mathbb{R}}^{0,2}$

Hence, $c_1(L(H,\alpha)) = ImH$.

Lemma 8.14. *If X is a complex torus we have*

$$\ker c_1 := Pic^0(X) \cong H^1(X,\mathscr{O})/H^1(X,\mathbb{Z}) \cong \overline{V}^*/\hat{\Gamma}$$

where $\hat{\Gamma} = \{l \in \overline{V}^* \mid Im(l(\Gamma)) \subset \mathbb{Z}\}$. That means that $Pic^0(X)$ is isomorphic to the group $Hom(\Gamma, U(1))$ of unitary characters defined in section 7.1. It is a torus and it is called dual torus of X.

Proof. We defined $Pic^0(X) := \ker c_1$. Hence for the exactness of the sequence we have

$$\ker c_1 \cong imH^1(e) \cong \overline{V}^*/\ker H^1(e) \cong \overline{V}^*/imH^1(i) \cong \overline{V}^*/\Gamma^*,$$

where the last equivalence holds since $H^1(i)$ is injective. Therefore we want to prove $H^1(i)(\Gamma^*) \cong \hat{\Gamma}$. A unitary character is completely determined by its behaviour on a basis of Γ: $\alpha(\gamma_i) = e^{2\pi i a_i}$. We associate to it the $\mathbb{R}$-linear form $l_{\mathbb{R}}\colon V \to \mathbb{R}$ such that $l_{\mathbb{R}}(\gamma_j) = a_j$ and then the $\mathbb{C}$-linear form $l(z) = -l_{\mathbb{R}}(iz) + il_{\mathbb{R}}(z)$ and hence we get a surjection $\overline{V}^* \to Hom(\Gamma, U(1))$ whose kernel is $\hat{\Gamma}$ because $e^{2\pi i a_i} = 1$ if and only if $a_i \in \mathbb{Z}$. Since $Hom(\Gamma, U(1)) \cong U(1)^{2g}$ is compact, $\hat{\Gamma}$ is a lattice. □

In the last part of this section we prove the most important theorem about line bundles on complex tori and study some properties of $Pic^0(X)$.

Lemma 8.15. *The following sequence is exact*

$$0 \longrightarrow Pic^0(X) \longrightarrow Pic(X) \xrightarrow{c_1} \Lambda^2\Gamma^* \cap \Lambda^{1,1}V^* \longrightarrow 0$$

Proof. Obvious by the definition of $Pic^0(X)$ and by the fact that c_1 is surjective due to Proposition 8.12. □

Theorem 8.16 (Appell-Humbert)**.** *Every line bundle over a complex torus is of the form $L = L(H,\alpha)$ and the pair (H,α) is uniquely determined.*

Proof. We know that $Pic^0(X)$ is the subgroup of $Pic(X)$ formed by all bundles of type $(0,\alpha)$. The exactness of the sequence in the lemma ensure us that c_1 is surjective and its image coincides with the Hermitian forms on V with imaginary part integer on Γ. We have to prove that for all $H \in \Lambda^2\Gamma^* \cap \Lambda^{1,1}V^*$ there exists $\alpha\colon \Gamma \to U(1)$ semicharacter for H. We can build it: let $\omega = ImH$ and define

$$\alpha(\gamma) = e^{-\frac{\pi}{2}H(\gamma,\gamma)}$$

That satisfies

$$\alpha(\gamma_1 + \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)e^{-\pi ReH(\gamma_1,\gamma_2)} = \alpha(\gamma_1)\alpha(\gamma_2)e^{i\pi i ReH(\gamma_1,\gamma_2)} = \alpha(\gamma_1)\alpha(\gamma_2)e^{i\pi\omega(\gamma_1,\gamma_2)}$$

Where the last equality holds because $i \cdot ReH(\gamma_1, \gamma_2) = i \cdot \omega(\gamma_1, i\gamma_2) = i \cdot (-i)\omega(\gamma_1, \gamma_2) = \omega(\gamma_1, \gamma_2)$. $\square$

Now that we discovered that every line bundle is of the form $L = L(H, \alpha)$ we can reformulate the notion of tensor product and inverse of line bundles:

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) \cong L(H_1 + H_2, \alpha_1\alpha_2) \quad \text{and} \quad L^{-1}(H, \alpha) \cong L(-H, \alpha^{-1})$$

Remark. This, with Proposition 8.13, implies an additive property of the first Chern class:

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

Remark. Lemma 8.15 is more general and it holds for any compact Kähler manifolds X in this form: the following sequence is exact

$$0 \longrightarrow Pic^0(X) \longrightarrow Pic(X) \xrightarrow{c_1} NS(X) \longrightarrow 0$$

where $NS(X)$ is the Néron-Severi group of X. Roughly speaking we can see $NS(X)$ as $Pic(X)/Pic^0(X)$, $Im(c_1)$ or $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$.

9 Interlude

9.1 Riemann Conditions

In the previous chapter we proved that if a torus can be embedded in a projective space, then it has an integer Kähler form. We will find an other condition involving the lattice (called Riemann Condition) for a torus to have such a Kähler form. In the next chapter we will see that a "very general" complex torus does not admit such an integer Kähler form. Let us recall the following

Definition. Given a bilinear alternating form ω integer on Γ, we call the Pfaffian of ω, $\mathrm{pf}(\omega)$, the square root of the determinant of the matrix representing ω.

The definition is well given because all the matrices representing ω have the same determinant in all different bases of Γ. In fact change of basis has determinant ± 1.

Proposition 9.1. *Let ω be a bilinear alternating integer form on Γ. There exist some positive integers $d_1, \dots, d_g$ such that $d_1 | \dots | d_g$ and a basis $\gamma_1, \dots, \gamma_{2g}$ of Γ such that the matrix associated with ω is*

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

where Δ is a diagonal matrix with entries $d_1, \dots, d_g$ which depend only by ω and hence $pf(\omega) = d_1 \cdots d_g$.

Proof. Let d_Γ the smallest positive value of ω and let $\gamma_1, \gamma_{g+1} \in \Gamma$ such that $\omega(\gamma_1, \gamma_{g+1}) = d_\Gamma$. For all $\gamma \in \Gamma$ the integer d_Γ divides $\omega(\gamma_1, \gamma)$ and $\omega(\gamma, \gamma_{g+1})$: indeed, if there exists $\bar{\gamma}$ such that $\mathrm{d}_\Gamma \nmid \omega(\gamma_1, \bar{\gamma})$ it follows that $MCD(\mathrm{d}_\Gamma, \omega(\gamma_1, \bar{\gamma})) = m < d_\Gamma$ and then there exist r and s in $\mathbb{Z}$ such that $m = r\omega(\gamma_1, \bar{\gamma}) + s d_\Gamma$. Hence

$$\omega(\gamma_1, r\bar{\gamma} + s\gamma_{g+1}) = \omega(\gamma_1, r\bar{\gamma}) + \omega(\gamma_1, s\gamma_{g+1}) = r\omega(\gamma_1, \bar{\gamma}) + s d_\Gamma = m < d_\Gamma$$

and it is a contradiction. It ensures that for all $\gamma \in \Gamma$

$$\eta = \gamma - \frac{\omega(\gamma_1, \gamma)}{d_\Gamma}\gamma_{g+1} - \frac{\omega(\gamma, \gamma_{g+1})}{d_\Gamma}\gamma_1 \in \Gamma.$$

Since η is ω-orthogonal to $\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_{g+1} = \langle \gamma_1, \gamma_{g+1} \rangle_{\mathbb{Z}}$, we have

$$\Gamma = (\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_{g+1}) \oplus (\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_{g+1})^\perp := (\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_{g+1}) \oplus \Gamma'.$$

Let $x, y \in \Gamma'$ such that $\omega(x, y) = d_{\Gamma'}$. Hence there exist $q, r \in \mathbb{Z}$ such that $d_{\Gamma'} = q d_\Gamma + r$ with $0 \le r < d_\Gamma$. Hence

$$\omega(x, y) = q\omega(\gamma_1, \gamma_{g+1}) + r \implies \omega(x, y) - q\omega(\gamma_1, \gamma_{g+1}) = r \implies r = \omega(x - q\gamma_1, y - \gamma_{g+1}) = 0$$

where the last equality holds because $d_{\Gamma'}$ is by definition the minimum. Hence d_Γ divides $d_{\Gamma'}$. We conclude by recurrence on the rank of Γ.
To show that $d_1, \dots, d_g$ depend only by ω it suffices to notice that the greatest common divisor of $2r$-minors of the matrix of ω is independent by the choice of the basis. ◻

Theorem 9.2 (Riemann Conditions)**.** *An integer Kähler form on a torus $X = V/\Gamma$ does exist if and only if there exists a complex basis $\mathcal{B}$ of V, some positive integers $d_1, \ldots, d_g$ such that $d_1 | \ldots | d_g$ and a square symmetric matrix τ whose imaginary part is definite positive such that:*

$$\Gamma = \tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g.$$

Proof. Using former notations, we notice that the vectors $e_j = \gamma_{g+j}/d_j$, $j = 1, \ldots, g$ forms a complex basis $\mathcal{B}$ for V. Let us denote with $(\tau\ \Delta)$ the components matrix of $\gamma_1, \ldots, \gamma_{2g}$ in this basis and set $R = \mathfrak{Re}\,\tau$ and $S = \mathfrak{Im}\,\tau$. The matrix of ω in components with respect to the real basis (e_j, ie_j) is

$${}^t\begin{pmatrix} R & \Delta \\ S & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \begin{pmatrix} R & \Delta \\ S & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & S^{-1} \\ -{}^tS^{-1} & {}^tS^{-1}(R - {}^tR)S^{-1} \end{pmatrix}.$$

So that ω is of type $(1,1)$ it suffices that $\omega(ix, iy) = \omega(x, y)$ for all x, y. By calculation we get that ${}^tS^{-1}(R - {}^tR)S^{-1} = 0$ and ${}^tS^{-1} = S^{-1}$, i.e. R and S must be symmetric. It implies that the matrix of the Hermitian form associated with ω in the basis $\mathcal{B}$ is S^{-1} and hence it is definite positive.

On the other side, if the lattice Γ has the form $\tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g$ in the basis $\mathcal{B}$, the Hermitian form H with matrix $(Im\tau)^{-1}$ is definite positive. Hence the matrix of $\omega = ImH$ is $\begin{pmatrix} 0 & S^{-1} \\ {}^tS^{-1} & 0 \end{pmatrix}$ in the basis $\mathcal{B}$ and $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$ in the basis of Γ given by the columns of the matrix $(\tau\ \Delta)$. Thus it is integer on Γ. □

We want to relate complex and symplectic structures of Γ and V. The basis as in Proposition 9.1 is also called symplectic basis of Γ for the Kähler (symplectic) form ω. Taking half of the basis leads to a direct sum decomposition $\Gamma = \Gamma_1 \oplus \Gamma_2 = \langle \gamma_1, \ldots, \gamma_g \rangle \oplus \langle \gamma_{g+1}, \ldots, \gamma_{2g} \rangle$ where Γ_i are isotropic (i.e. $\Gamma_i \subseteq \Gamma_i^{\perp_\omega}$). If we denote by W the vector subspace $\langle \gamma_1, \ldots, \gamma_g \rangle_{\mathbb{R}}$, it is clear that there is a direct sum decomposition $V = W \oplus iW$ where, by the linearity of ω, both W and iW are maximal isotropic.

Remark. All decompositions $\Gamma = \Gamma_1 \oplus \Gamma_2$ into maximal ω-isotropic subspaces lead to a decomposition of V in maximal ω-isotropic subspaces. The converse is no more true: not always a decomposition in maximal ω-isotropic subspaces of V leads to a maximal ω-isotropic subspaces of Γ. For example it happens when the maximal isotropic subspace W does not intersect "well" Γ, in the sense that $\mathrm{rank}(W \cap \Gamma) < g$.

The above remark allows us to better understand the classes in $Pic(X)/Pic^0(X)$

$$Pic^H(X) := \{L(H, \alpha) \mid \alpha \text{ semicharacter for } H\}$$

In fact if $H \in NS(X)$ is non degenerate (and hence so is $\omega = imH$), a decomposition $V = V_1 \oplus V_2$ leads to define the map

$$\begin{array}{cccc} \alpha_0 \colon & V & \longrightarrow & U(1) \\ & v = v_1 + v_2 & \mapsto & e^{\pi i \omega(v_1, v_2)} \end{array}$$

And by an obvious computation, using that V_i are isotropic, we have the following.

Lemma 9.3. *For every $v = v_1 + v_2, w = w_1 + w_2 \in V_1 \oplus V_2$,*

$$\alpha_0(v + w) = \alpha_0(v)\alpha_0(w)e^{\pi i \omega(v,w)}e^{-2\pi i \omega(v_2, w_1)}.$$

Hence we can consider the line bundle $L_0 := L(H, \alpha_0)$ uniquely determined by H and by the chosen decomposition. In other world the decomposition in maximal ω-isotropic subspaces of V distinguish a canonical representative $L_0 \in Pic^H(X)$.

Proposition 9.4. *Suppose H is a non degenerate Hermitian form on V and let $V_1 \oplus V_2$ a ω-decomposition.*

- *L_0 is the unique line bundle in $Pic^H(X)$ whose semicharacter is is trivial on $\Gamma_i = V_i \cap \Gamma$.*
- *For every $L = L(H, \alpha) \in Pic^H(X)$ there is a point $c \in V$ such that $L \cong \tau_{\bar{c}}^* L_0$*

Proof. The first assertion follows directly from the definition of α_0 since $\omega_{|\Gamma_i} = 0$.
The second one follows from Proposition 8.9. □

Up to now we used an approach consisting in "choosing the right basis" for V and Γ and see what happens. It helps a lot in understanding how the presence of an integer Kähler form affects the lattice, but it seems to me that the converse is less clear. We propose below an approach that do not fix any basis of V nor of Γ based on the period matrix.

Definition. Given a torus $X = V/\Gamma$, choose a basis $(e_1, \ldots, e_g)$ of V and $(\gamma_1, \ldots, \gamma_{2g})$ of Γ. We can write any γ_i in terms of the e_j in this way: $\gamma_i = \sum_j \gamma_{i,j} e_j$. The matrix

$$\Pi = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,2g} \\ \vdots & & \vdots \\ \gamma_{g,1} & \cdots & \gamma_{g,2g} \end{pmatrix}$$

is called period matrix for X and it determines completely it.

Remark. With respect to these basis it is clear that

$$X = \mathbb{C}^g / \Pi \mathbb{Z}^{2g}.$$

We ask ourselves: what it means a complex torus is an abelian variety in terms of period matrix?

Theorem 9.5. *X is an abelian variety if and only if there is a non degenerate alternating matrix $A \in M_{2g}(\mathbb{Z})$ such that*

(1) $\Pi A^{-1}\, {}^t\Pi = 0$

(2) $i \Pi A^{-1}\, {}^t\overline{\Pi} > 0$

These are the Riemann Conditions. It turns out that A is the matrix of the alternating form ω defining the integer Kähler form.
The proof of the theorem follows straightforward from the following two lemmas.

Lemma 9.6. *Let H the bilinear form associated with ω. H is Hermitian if and only if $\Pi A^{-1}\, {}^t\Pi = 0$*

Proof. We recall that H is Hermitian if and only if the form ω is of type (1,1). The rest is a straightforward calculation. See (BL) Lemma 4.2.2. □

Lemma 9.7. *Suppose H is Hermitian. Then the matrix $2i(\overline{\Pi} A^{-1}\, {}^t\Pi)^{-1}$ is the matrix of H with respect to the basis $(e_1, \ldots, e_g)$. In particular H is definite positive if and only if $i \Pi A^{-1}\, {}^t\overline{\Pi} \geq 0$.*

Proof. See (BL) Lemma 4.2.3. □

We want to bind these approaches: let (X, ω) be an abelian variety. Fix a symplectic basis $(\gamma_1, \dots, \gamma_{2g})$ for Γ such that ω has matrix $A = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$. The period matrix with respect to a generic complex basis $(e_1, \dots, e_g)$ is then of the form $\Pi = (\Pi_1, \Pi_2)$. We can rephrase the Riemann Condition this way

(1) $\Pi_2 \Delta^{-1}\,{}^t\Pi_1 - \Pi_1 \Delta^{-1}\,{}^t\Pi_2 = 0$

(2) $i\Pi_2 \Delta^{-1}\,{}^t\overline{\Pi}_1 - i\Pi_1 D^{-1}\,{}^t\overline{\Pi}_2 > 0$

If moreover we take the basis $(e_1, \dots, e_g) = (\gamma_1/d_1, \dots, \gamma_g/d_g)$, we get $(\Pi_1, \Pi_2) = (\tau, \Delta)$ and the relations become

(1) $\Pi_2 \Delta^{-1}\,{}^t\Pi_1 - \Pi_1 \Delta^{-1}\,{}^t\Pi_2 = \Delta\Delta^{-1}\,{}^t\tau - \tau\Delta^{-1}\,{}^t\Delta = {}^t\tau - \tau = 0$

(2) $i\Pi_2 \Delta^{-1}\,{}^t\overline{\Pi}_1 - i\Pi_1 \Delta^{-1}\,{}^t\overline{\Pi}_2 = i\Delta\Delta^{-1}\,{}^t\overline{\tau} - i\tau\Delta^{-1}\,{}^t\overline{\Delta} = i({}^t\overline{\tau} - \tau) = i(\overline{\tau} - \tau) = i(-2i\,Im\tau) = 2Im\tau > 0$

That are exactly the Riemann Condition we found in Theorem 9.2.

9.2 The Dual Complex Torus $Pic^0(X)$

In this section we make some consideration about $Pic^0(X)$ that, thanks to Lemma 8.14, has the structure of a complex torus and it is isomorphic to the group $Hom(\Gamma, U(1))$. First of all we can read this isomorphism in light of Appel-Humbert theorem. In fact it states that any line bundle is of the form $L = L(H, \alpha)$ and since $Pic^0(X)$ is defined as $\ker c_1$, it follows that it is completely defined by the semicharacters α for $H = 0$, that are exactly the elements in $Hom(\Gamma, U(1))$.

Definition. The group $Pic^0(X)$ is a torus of the same dimension of X and is called the dual complex torus of X. We will also denote it by $\hat{X}$.

Remark. The name "dual torus" comes from the fact that $\Gamma^* \subseteq \overline{V}^*$ is actually the dual lattice of $\Gamma \subseteq V$, referring to the notation of Proposition 8.14. This leads to

$$\hat{\hat{X}} = X.$$

We would like to establish a formal relation between X and $\hat{X}$. Recall that an isogeny is a map between complex tori that is a surjective morphism with finite kernel. The cadinality of the kernel is the degree of the isogeny. We notice that if $u\colon X \to Y$ is a isogeny (in particular a group morphism) the kernel is a finite subgroup of X, hence there exists an integer number $n > 0$ such that $n(\ker u) = \{0\}$. This implies that the morphism of multiplication by n, namely $\mathbf{n}\colon Y \to Y$ factorises in $\mathbf{n}\colon Y \xrightarrow{v} X \xrightarrow{u} Y$ where v is an isogeny. From here follows the proof that "to be isogenic" is an equivalence relation (see Proposition 1.4).
In the following theorem we will prove that there is an isogeny between X and $\hat{X}$.

Theorem 9.8. *Let X be a complex torus and $L = L(H, \alpha)$ a line bundle over X. The morphism*

$$\begin{aligned} \varphi_L\colon X &\longrightarrow \hat{X} \\ x &\mapsto \tau_x^* L \otimes L^{-1} \end{aligned}$$

does not depend by anything else but $c_1(L)$. Its kernel, denoted by $K(L)$, is a disjoint union of subtori of X of the same dimension of $\ker\omega$. *If L is not degenerate, φ_L is an isogeny of degree $pf(\omega)^2$, where ω is the bilinear alternating form associated to H, i.e. $\omega = imH$.*

Proof. Let ω be a representative for $c_1(L)$ and $x \in X$. We will denote $\tilde{x}$ a point in $\pi^{-1}(x) \subseteq V$. For the Lemma 8.9, $\varphi_L(x)$ is the class of the line bundle $L(0, e^{2i\pi\omega(\tilde{x},\cdot)})$. Hence φ_L is a group morphism thanks to the Theorem of the Square, has values in $\hat{X}$ and it depens only by $\omega = c_1(L)$.
Moreover

$$K(L) = \{x \in X \mid \omega(\gamma, \tilde{x}) \in \mathbb{Z},\ \forall \gamma \in \Gamma\}$$

is the finite union of tranlsations of $\ker\omega \subseteq X$. Notice that is well defined because if we pick a different $\hat{x} \in \pi^{-1}(x)$ it means that $\tilde{x} = \hat{x} + \hat{\gamma}$ and $\omega(\gamma, \hat{\gamma}) \in \mathbb{Z}$. In order to prove that they are subtori, we recall that, since ω is non degenerate, there exist a basis of Γ such that the matrix of ω has the form

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

where Δ is a diagonal matrix with entries $d_1, \dots, d_g$ which depend only by ω. The group $K(L)$ is then the image in V/Γ of $\langle \gamma_1/d_1, \dots, \gamma_g/d_g, \gamma_{g+1}/d_1, \dots, \gamma_{2g}/d_g\rangle_{\mathbb{Z}} \subseteq \Gamma$ and hence isomorphic to $(\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z})^2$ thus of cardinality $(d_1 \cdots d_g)^2 = \mathrm{pf}(\omega)^2$. □

Lemma 9.9. *If $X = V/\Gamma$ and $\hat{X} = \bar{V}^*/\hat{\Gamma}$, the analytic representation of φ_L is*

$$\varphi_H : V \longrightarrow \bar{V}^*$$
$$v \mapsto H(v, \cdot)$$

Proof. Direct consequence of Prop 8.9. □

Corollary 9.10. *The following hold:*

(1) $\varphi_{L\otimes M} = \varphi_L + \varphi_M$ for all $L, M \in Pic(X)$.

(2) $\varphi_L = \hat{\varphi}_L$ under the identification $\hat{\hat{X}} = X$.

Proof. It follows from the definition of φ_L. □

In the previous theorem we defined $K(L)$ as the kernel of φ_L. In order to describe this object, define

$$\Gamma(L) = \{v \in V \mid \omega(v, \Gamma) \subset \mathbb{Z}\}.$$

Obviously $\Gamma(L) = \varphi_H^{-1}(\Gamma^*)$, and hence

$$K(L) = \Gamma(L)/\Gamma.$$

It has the following properties.

Proposition 9.11. *For any line bundle L on X*

(1) $K(L \otimes P) = K(L)$ for all $P \in Pic^0(X)$.

(2) $K(L) = X$ if and only if $L \in Pic^0(X)$.

(3) $K(L^n) = n_X^{-1}K(L)$ *for any* $n \in \mathbb{Z}$.

(4) $K(L) = n_X K(L^n)$ *for any* $n \in \mathbb{Z}, n \neq 0$.

Proof. *(1)* and *(2)* follow from the fact that $c_1(L) = 0$ if and only if $L \in Pic^0(X)$. We conclude by the remark to Lemma 8.9. *(3)* and *(4)* are a consequence of the fact that $L(H,\alpha)^{\otimes n} = L(nH, \alpha^n)$ and thus $\Gamma(L) = \{v \in V \mid ImH(nv, \Gamma) \subseteq \mathbb{Z}\} = \{\frac{1}{n}v \in V \mid v \in \Gamma(L)\}$. □

Let us give the following

Definition. A line bundle L over X is non degenerate if his alternating form $\omega = imH = c_1(L)$ is non degenerate.

The non degeneracy of L is strictly connected with $K(L)$.

Proposition 9.12. *A line bundle L is non degenerate if and only if* $K(L)$ *is finite.*

Let us introduce a bilinear alternating form

$$\begin{aligned} e^L \colon K(L) \times K(L) &\longrightarrow \mathbb{C}^* \\ (x, y) &\longmapsto e^{2\pi i \omega(\tilde{x}, \tilde{y})} \end{aligned}$$

Proposition 9.13. e^L *is non degenerate if and only if L is non degenerate.*

Proof. If L is non degenerate, $\omega \neq 0$ and let us consider K_1, the soubgroup of $K(L)$ spanned by the images of $\gamma_1/d_1, \dots, \gamma_g/d_g$ and K_2 by $\gamma_{g+1}/d_1, \dots, \gamma_{2g}/d_g$. It holds that $K(L) = K_1 \oplus K_2$ and K_i is an isotropic subspace with respect to e^L and hence it induces an isomorphism $K_1 \cong K_2^*$ and thus it is non degenerate.
If L is degenerate, the kernel of φ_H is different from $\{0\}$ and its image in X is contained in the kernel of e^L. □

9.3 The Riemann-Roch Theorem

In the previous sections we occasionally talked about projectivity of complex tori and we discovered it is strictly related to the existence of an integer Kähler form. Actually projectivity of a compact complex manifold is also related to the existence of very ample line bundles. We will define what very ample line bundle is in the following sections, but it is important to know that it is characterised by having a lot of global sections. We will present now the Riemann-Roch theorem: a powerful tool that will allow us to relate the Kähler form associated with a line bundle with the number of its global sections. Remark that, as we saw in Proposition 8.6, it is equivalent to count the number of theta functions of the same type of the bundle.

Let L a line bundle and $V = W \oplus iW$ a decomposition of maximal isotropic subspaces for $\omega = ImH = c_1(L)$. The Hermitian form H is real on $W \times W$ since its imaginary part is null there, hence it extends to a $\mathbb{C}$-linear symmetric form B on the whole $V \times V$.

Lemma 9.14. *It holds that*

$$(H - B)(x, y) = \begin{cases} 0 & \textit{if } x \in W \\ 2i\omega(x, y) & \textit{if } y \in W \end{cases}$$

Proof. We know that $H - B = 0$ on $W \times W$, hence if $x \in W$ and $H - B$ is $\mathbb{C}$-linear on the second variable we have that

$$(H - B)(x, y) = (H - B)(x, w_1 + iw_2) = (H - B)(x, w_1) + i(H - B)(x, w_2) = 0$$

as desired. Moreover if $y \in W$:

$$(H - B)(x, y) = (\overline{H} - B)(y, x) = (\overline{H} - H)(y, x) = -2i\omega(y, x) = 2i\omega(x, y).$$

□

Theorem 9.15 (Riemann-Roch)**.** *Let X be a complex torus and L a line bundle whose first Chern class is definite positive. It holds that*

$$dim H^0(X, L) = pf\big(c_1(L)\big) > 0$$

Proof. Thanks to Appel-Humbert theorem it suffices to prove it in the case $L = L(H, \alpha)$. We can then rephrase the thesis and compute the dimension of the vector space of holomorphic functions θ on V satisfying

$$\theta(z + \gamma) = \alpha(\gamma) e^{\pi H(\gamma, z) + \frac{\pi}{2} H(\gamma, \gamma)} \theta(z).$$

We remark that it suffices to prove it on the basis $(\gamma_1, \dots, \gamma_{2g})$ of Γ as in 9.1. This allows us to take the homomorphism given by the restriction of α to $\Gamma' = \langle \gamma_1, \dots, \gamma_g \rangle$. Therefore there exists a $\mathbb{C}$-linear form l on V such that $\alpha(\gamma) = e^{2i\pi l(\gamma)}$ for all $\gamma \in \Gamma'$. Let us define

$$\tilde{\theta}(z) = e^{-\frac{\pi}{2} B(z,z) - 2i\pi l(z)} \theta(z).$$

It is a theta function of the same type as θ and verifies

$$\tilde{\theta}(z + \gamma) = \alpha(\gamma) e^{-2i\pi l(\gamma)} e^{\pi (H-B)(\gamma, z) + \frac{\pi}{2}(H-B)(\gamma,\gamma)} \tilde{\theta}(z).$$

It is hence Γ'-periodic and hence we can consider its Fourier series expansion

$$\tilde{\theta}(z) = \sum_{m \in \mathbb{Z}^g} c(m) e^{2i\pi \sum_k m_k z_k},$$

where $z = \sum_{k=1}^{g} z_k \gamma_k$, with $z_1, \dots, z_g \in \mathbb{C}$. Moreover it verifies

$$\tilde{\theta}(z + \gamma_{g+j}) = b_j e^{\pi (H-B)(\gamma_{g+j}, z)} \tilde{\theta}(z),$$

where $b_j = \alpha(\gamma_{g+j}) e^{-2i\pi l(\gamma_{g+j})} e^{(H-B)(\gamma_{g+j}, \gamma_{g+j})}$ is a non null constant. The previous lemma ensure us that

$$(H - B)(\gamma_{g+j}, z) = \sum_k z_k (H - B)(\gamma_{g+j}, \gamma_k) = -2i d_j z_j.$$

One has $\gamma_{g+j} = \sum_{k=1}^{g} \frac{\gamma_k}{d_k} \tau_{kj}$, and hence $z + \gamma_{g+j} = \sum_{k=1}^{g} \left(\frac{\tau_{kj}}{d_k} + z_k \right) \gamma_k$. Putting together this results one gets

$$\sum_{m \in \mathbb{Z}^g} c(m) e^{2\pi i (\sum_k m_k (\frac{\tau_{kj}}{d_k} + z_k))} = b_j \sum_{m \in \mathbb{Z}^g} c(m) e^{2\pi i (\sum_k m_k z_k - d_j z_j)}$$

for all $z_1, \dots, z_g \in \mathbb{C}$. It follows that, for the uniqueness of the Fourier expansion,

$$c(m) e^{2\pi i (\sum_k \frac{m_k}{d_k} \tau_{kj})} = b_j c(m + d_j \epsilon_j)$$

for all $m \in \mathbb{Z}^g$ and $j \in \{1, \ldots, g\}$, where ϵ_j is the zero vector with a one in the coordinate j. This shows that the coefficients $c(m)$, and hence θ, are determined by those for which $0 \leq m_j \leq d_j$ for all j. In particular:

$$dim H^0(X, L) \leq d_1 \cdots d_g = pf(\omega).$$

Conversely, if are given complex numbers $c(m)$ such that $0 \leq m_j \leq d_j$ for all j, we can define the coefficients $c(m)$ for all m by the relation

$$c(m) e^{2\pi i (\sum_k \frac{m_k}{d_k} \tau_{kj})} = b_j c(m + d_j \epsilon_j).$$

By recurrence one verify that

$$|c(m)| \leq \left| e^{2i\pi \sum_{i,j} \frac{m_j}{d_j} \frac{m_k}{d_k} \tau_{jk} + o(m)} \right|$$

and hence the Fourier series converges, since $Im \tau$ is definite positive. Therefore we get a theta function and hence a section of L. $\square$

We gave a proof just for the case $c_1(L)$ is definite positive. This is the most interesting case for us because it includes the case where $c_1(L)$ is an integer Kähler form. Anyway there is a more complete version of this theorem that, other than consider not only the positive definite case, uses all cohomology groups.

Theorem 9.16 (Riemann-Roch, extended version). *Let L be a line bundle on X, whose Hermitian form associated with the first Chern class ω has s negative eigenvalues. Then*

$$\chi(L) := \sum_{j=0}^{g} (-1)^j dim H^j(X, L) = (-1)^s pf(\omega)$$

Proof. See (BL),Theorem 3.6.1. $\square$

10 Abelian Varieties

Abelian varieties are those complex tori which admit an embedding in a projective space (we say they are projective). We have already seen in Section 5 that a necessary condition to be projective is to have an integer Kähler form, indeed any projective manifold admits an integer Kähler form given by the restriction of the Fubini-Study form of $\mathbb{P}^n$. In Section 9.1 we proved the Riemann Conditions for the existence of an integer Kähler form on a complex torus.

In this chapter we will see an another way to establish if a torus is projective using the notion of very ample line bundles and the Kodaira embedding theorem.

10.1 Line Bundles on the Projective Space

Before going on with complex tori we should study line bundles on the projective space. Let W be a complex vector space. The most näive idea to build a line bundle $q\colon L \longrightarrow \mathbb{P}W$ is to glue on each $x \in \mathbb{P}W$ the line $l_x = \langle x \rangle \subseteq W$:

$$L = \{(x, v) \in \mathbb{P}W \times W \mid v \in l_x\}.$$

If $\{U_\alpha\}$ is the standard cover for $\mathbb{P}W$ with respect to the canonical basis $\mathcal{B}$ of W, we can build a trivializing athlas and its transition maps. Set $\psi_\alpha\colon q^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ defined by $\psi_\alpha(x, v) = (x, v_\alpha)$, where $v_\alpha \in \mathbb{C}$ is the α-th coordinate of the vector v with respect to the basis $\mathcal{B}$. Notice that it is well defined because $x \in U_\alpha$ implies $x_\alpha \neq 0$ and by our construction $v_\alpha = \mu x_\alpha$ for some $\mu \in \mathbb{C}$.

In $U_\alpha \cap U_\beta$ we have, for a generic $\lambda \in \mathbb{C}$,

$$\psi_\alpha \circ \psi_\beta^{-1}(x, \lambda) = \psi_\alpha\Big(x, \frac{\lambda}{x_\beta} x\Big) = \Big(x, \lambda \frac{x_\alpha}{x_\beta}\Big)$$

and hence $g_{\alpha\beta}(x) = \frac{x_\alpha}{x_\beta} \in GL(1, \mathbb{C})$. We denote this line bundle by $\mathcal{O}_{\mathbb{P}W}(-1)$ and by $\mathcal{O}_{\mathbb{P}W}(1)$ its dual line bundle.

Definition. For all positive integers k we set

$$\mathcal{O}_{\mathbb{P}W}(k) := \mathcal{O}_{\mathbb{P}W}(1)^{\otimes k} \quad \text{and} \quad \mathcal{O}_{\mathbb{P}W}(-k) := \mathcal{O}_{\mathbb{P}W}(-1)^{\otimes k}.$$

Proposition 10.1. *All line bundles over $\mathbb{P}W$ are of this type, hence $Pic(\mathbb{P}W) \cong \mathbb{Z}$.*

Proposition 10.2. *The vector space of section of $\mathcal{O}_{\mathbb{P}W}(k)$ is null for all $k < 0$ and it is isomorphic to the vector space of homogeneous polynomial of degree k for $k \geq 0$. In particular the space of section of $\mathcal{O}_{\mathbb{P}W}(1) \cong W^*$.*

Proof. See (Huy), Proposition 2.4.1. □

Let us come back to complex tori. We would like to study the first Chern class of $\mathcal{O}_{\mathbb{P}W}(-1)$. As in Proposition 8.12 we have to define a metric and a section. Let us set on the fibre l_x the metric $||v||^2 = \sum_{j=1}^n v_j \overline{v}_j$, and the local section $s_\alpha\colon U_\alpha \to U_\alpha \times \mathbb{C}$, $s_\alpha(x) = x/x_\alpha$. Hence $c_1(\mathcal{O}_{\mathbb{P}W}(-1))$ is represented by

$$\frac{1}{2i\pi}\partial\overline{\partial} \log ||x/x_\alpha||^2 = \frac{1}{2i\pi}\partial\overline{\partial} \log ||x||^2 - \frac{1}{2i\pi}\partial\overline{\partial} \log ||x_\alpha \overline{x}_\alpha||^2 = \frac{1}{2i\pi}\partial\overline{\partial} \log ||x||^2$$

that is exactly $-\omega_{FS}$, thus

$$[\omega_{FS}] = -c_1(\mathcal{O}_{\mathbb{P}W}(-1)) = c_1(\mathcal{O}_{\mathbb{P}W}(1)).$$

It is instructive to see how the following diagram appears for the compact complex manifold $\mathbb{P}^n$:

$$\begin{tikzcd}
 & 0 \arrow[r] & \cancel{H^1(\mathbb{Z})} \arrow[r] & \cancel{H^1(\mathcal{O})} \arrow[dr, bend left] & & & \\
0 \arrow[r] & H^0(\mathcal{O}^*) \arrow[r] & H^0(\mathcal{M}^*) \arrow[r] & Div(\mathbb{P}^n) \arrow[r, "\Phi"] \arrow[drr, dotted, bend right, "\eta"] & Pic(\mathbb{P}^n) \arrow[r] \arrow[dr, bend right, "c_1"] & H^1(\mathcal{M}^*) \arrow[r] & \dots \\
 & & & & & \mathbb{Z} \cdot [\omega_{FS}] \arrow[r] & \dots
\end{tikzcd}$$

In conclusion, there is an injective map $Pic(\mathbb{P}^n) \to H^2(\mathbb{P}^n, \mathbb{Z})$ that, for dimensional reasons, is also surjective and hence

$$Pic(\mathbb{P}^n) \cong \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}W}(-1)$$

and proving Proposition 10.1.

10.2 Projectivity, Ample Line Bundles and Sections

In this section we would like to study when there exists a holomorphic embedding from a complex torus to a projective space. Theorem 5.2 gives us one condition related to Kähler geometry, now we are interested in something that involves line bundles and their sections.

Let $u\colon X \to \mathbb{P}W$ a holomorphic function from a torus to a projective space. We want someway to relate it to line bundles. In fact this map allows us to consider on X the pull-back line bundle $L = u^*\mathcal{O}_{\mathbb{P}W}(1)$ and it induces a linear application between vector spaces of sections

$$\Gamma(u)\colon W^* \cong \Gamma(\mathbb{P}W, \mathcal{O}_{\mathbb{P}W}(1)) \to \Gamma(X, L)$$

where the isomorphism $W^* \cong \Gamma(\mathbb{P}W, \mathcal{O}_{\mathbb{P}W}(1))$ holds thanks to Proposition 10.2.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, bend right, "\Gamma(u)(s)"'] & \mathbb{P}W \arrow[d, bend left, "s"] \\
L = u^*(\mathcal{O}(1)) \arrow[u] & \mathcal{O}(1) \arrow[l] \arrow[u]
\end{tikzcd}$$

Therefore a section s of $\mathcal{O}_{\mathbb{P}W}(1)$ is a linear function and hence it vanishes on a hyperplane. It implies that its image under $\Gamma(u)$ is null if and only if $u(X)$ is contained in this hyperplane. Thus, if $u(X)$ is not contained in any hyperplane, $\Gamma(u)$ is an injective function.

We can make a further step: we can pull-back the Kähler form $u^*\omega_{FS}$ on X and, since $[\omega_{FS}] = c_1(\mathcal{O}_{\mathbb{P}W}(1))$ and Chern classes behave well with pull-backs, it is the first Chern class of $L = u^*\mathcal{O}_{\mathbb{P}W}(1)$. By the Appel-Humbert theorem L is a line bundle of the form $L(H, \alpha)$, and hence $[u^*\omega_{FS}] = [imH]$ is an integer Kähler form on X.

Up to now we have seen how to build a line bundle starting from a holomorphic map $u\colon X \to \mathbb{P}W$. On the contrary, if we have a line bundle $L \to X$ we can consider the (finite dimensional) vector space of sections $\Lambda := \Gamma(X, L)$ of basis $(s_1, \dots, s_r)$ and we can define a meromorphic map onto a projective space

$$\begin{aligned} \Psi_L\colon X &\longrightarrow \mathbb{P}\Lambda \\ x &\mapsto (s_1(x) : \cdots : s_r(x)) \end{aligned}$$

Thanks to Proposition 8.6 and by the fact that $\Gamma(X, L) = H^0(X, L)$, we can say

$$\begin{aligned} \Psi_L \colon X &\longrightarrow \mathbb{P}\Lambda \\ x &\mapsto (\theta_1(\bar{x}) : \cdots : \theta_r(\bar{x})) \end{aligned}$$

where $\pi(\bar{x}) = x$ and θ_j are theta functions of the same type of L. We wonder which properties L must have to make this map holomorphic or an embedding. First of all we notice it is not well defined in points $x \in X$ such that all sections vanishes.

Definition. Let L be a line bundle on X. A point $x \in X$ is a base point of L if $s(x) = 0$ for all section $s \in H^0(X, L)$. We denote by $Bs(L)$ the set of all base points of L.

Remark. If we choose a basis $(s_1, \dots, s_r)$, $Bs(L) = Z(s_1) \cap \cdots \cap Z(s_r)$ is an analytic subvariety of X.

Remark. Let us denote by $|L| \subseteq Div(X)$ the set of all effective divisors defined as zero set of holomorphic sections of L, that is the image of the map $\mathbb{P}\Gamma(X, L) \to Div(X)$. Hence Ψ_L is not defined in $x \in X$ if and only if x is contained in every divisor in $|L|$. In fact $Bs(L)$ coincides with the intersection of all divisors in $|L|$.

Proposition 10.3. *The following map is well defined and holomorphic.*

$$\begin{aligned} \Psi_L \colon X \backslash Bs(L) &\to \mathbb{P}\Lambda \\ x &\longmapsto (\theta_1(\bar{x}) : \cdots : \theta_r(\bar{x})) \end{aligned}$$

Proof. See (Huy), Proposition 2.3.26. □

When Ψ_L is holomorphic, we can apply Proposition 7.11 to say that $\omega = ImH$ is an integer Kähler form on X.
The most incredible fact is that this two constructions are one the inverse of the other in the sense that, if we apply the previous construction to the map Ψ_L, we get the line bundle L, i.e.

$$L = \Psi_L^*(\mathscr{O}(1)) \quad \text{and} \quad [\omega] = [\Psi_L^* \omega_{FS}].$$

We want now Ψ_L to be an embedding. Recall that any map $u \colon X \to \mathbb{P}^n$ fits in a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\tilde{u}"] \arrow[d, "\pi"'] & \mathbb{C}^{n+1} \setminus \{0\} \arrow[d, "p"] \\ X \arrow[r, "u"] & \mathbb{P}^n \end{tikzcd}$$

and u is holomorphic if and only if $\tilde{u}$ is. Moreover we can add the fact that $\tilde{u} = (\theta_0, \dots, \theta_n)$. For u to be an embedding it suffices that it is injective and the following matrix has maximal rank $g+1$ for all z (implicit function theorem).

$$\begin{pmatrix} \theta_0 & \frac{\partial \theta_0}{\partial z_1}(z) & \dots & \frac{\partial \theta_0}{\partial z_g}(z) \\ \vdots & \vdots & & \vdots \\ \theta_n & \frac{\partial \theta_n}{\partial z_1}(z) & \dots & \frac{\partial \theta_n}{\partial z_g}(z) \end{pmatrix} \tag{1}$$

Hence it is clear that we have to study relations between line bundles and theta functions.

Corollary 10.4 (to the Theorem of the Square). *If θ is a normalized theta function associated with $L = L(H, \alpha)$ and $a_1, \dots, a_r$ are points of V such that $\sum a_i = 0$, the function*

$$z \mapsto \theta(z + a_1) \cdots \theta(z + a_r)$$

is a normalized theta function associated with the line bundles $L^r := L^{\otimes r}$.

Proof. Since it holds that $\theta \circ \tau_a(z) = \theta(z + a) = \theta(z) e^{2i\pi\omega(\cdot, a)}$,

$$\theta(z + a_1) \cdots \theta(z + a_r) = \theta(z)^r e^{2i\pi \sum \omega(\cdot, a_i)} = \theta(z)^r e^{2i\pi\omega(\cdot, \sum a_i)} = \theta(z)^r.$$

It is now clear that it is a theta function of type (rH, α^r). Moreover, thanks to the theorem of the square,

$$L \cong \tau_0^* L \cong \tau_{a_1+\dots+a_r}^* L \cong \tau_{a_1+\dots+a_{r-1}}^* L \otimes \tau_{a_r}^* L \otimes L^{-1} \cong \dots \cong \tau_{a_1}^* L \otimes \tau_{a_2}^* L \otimes \dots \otimes \tau_{a_r}^* L \otimes L^{-r+1}$$

□

We can now prove the most important theorem of this section.

Theorem 10.5 (Lefschetz). *Let X be a complex torus and L a line bundle over X.*

(1) If L has a non null section, the application Ψ_{L^r} defines a holomorphic function from X to a projective space for all $r \geq 2$.

(2) If the first Chern class of L is definite positive, the application Ψ_{L^r} defines an embedding from X to a projective space for all $r \geq 3$.

We will give a straightforward proof without the help of Proposition 10.3.

Proof. The main ingredient of the proof is the Theorem of the Square (8.10).
(1) Let θ a normalized theta function corresponding to a non null section of L. If $r \geq 2$ there exists for every $z_0 \in V$ a point $a \in V$ such that the quantity

$$\theta(z_0 - a)\theta(z_0 + (r-1)a)$$

is not null. Thanks to the previous corollary, the function

$$z \mapsto \theta(z - a)^{r-1}\theta(z + (r-1)a)$$

corresponds to a section of L^r that is not null in z_0, hence for every point in V there is at least one section non vanishing there and that is equivalent to say that the map Ψ_L is everywhere well defined and holomorphic.
(2) If $c_1(L)$ is positive definite, Riemann-Roch Theorem ensures us the existence of a non null section and hence for (1) we know that Ψ_L is holomorphic. Let us denote by θ the theta function associated with this section. Let $(\theta_0, \dots, \theta_n)$ a basis for $H^0(X, L^r)$ and z_0 a point in V. The proof goes on by contradiction to 1: let us suppose that there exist complex numbers $\lambda_0, \dots, \lambda_g$ such that for all $j \in \{0, \dots, n\}$

$$\lambda_0 \theta_j(z_0) + \sum_{k=1}^{g} \lambda_k \frac{\partial \theta_j}{\partial z_k}(z_0) = 0.$$

For the theorem of the square, the function

$$\theta_{ab}(z) = \theta(z-a)^{r-2}\theta(z-b)\theta(z+(r-2)a+b)$$

is a normalized theta function corresponding to a section of L^r for all a and b in V. Thus it is a linear combination of θ_i and hence

$$\lambda_0\theta_{ab}(z_0) + \sum_{k=1}^{g} \lambda_k \frac{\partial \theta_{ab}}{\partial z_k}(z_0) = 0.$$

The relation

$$\psi(z) = \sum_{k=1}^{g} \lambda_k \frac{\partial \log \theta}{\partial z_k}(z)$$

defines a meromorphic function on V that verifies

$$(r-2)\psi(z_0-a) + \psi(z_0-b) + \psi(z_0+(r-2)a+b) = \lambda_0$$

for all a and b in V. If we fix a_0, there always exists a b such that $\theta(z_0-b)\theta(z_0+(r-2)a_0+b) \neq 0$ and hence the functions $a \mapsto \psi(z_0-b)$ and $a \mapsto \psi(z_0+(r-2)a+b)$ are holomorphic in a neighbourhood of a_0. If $r \geq 3$, ψ is holomorphic in a neighbourhood of $z_0 - a_0$ and hence holomorphic on V since a_0 was an arbitrary point. It holds that

$$\psi(z+\gamma) = \pi H(\gamma, \lambda) + \psi(z)$$

where H is the Hermitian form associated with $c_1(L)$ and $\lambda = (\lambda_1, \dots, \lambda_g) \in V$. It follows that partial derivatives of ψ are Γ-periodic and hence constant, thus ψ is an affine function. Moreover $\psi(\gamma)-\psi(0) = \pi H(\gamma,\lambda)$ and hence they are $\mathbb{R}$-linear and $\psi(z)-\psi(0) = \pi H(z,\lambda)$. This is a contradiction since the RHS is $\mathbb{C}$-antilinear and the LHS is $\mathbb{C}$-linear, hence $H(z,\lambda) = 0$ for all z and, since $c_1(L)$ is non degenerate, it implies $\lambda = 0$. □

Definition. A line bundle L over X is very ample if the morphism Ψ_L is an embedding.

Definition. A line bundle L over X is ample if there exist an integer $r > 0$ such that L^r is very ample.

Definition. We say that a line bundle is positive if its first Chern class is definite positive.

Lefschetz theorem provides a proof of the Kodaira embedding theorem in the case of complex tori.

Theorem 10.6 (Kodaira)**.** *Let X be a compact Kähler manifold and L a holomorphic line bundle on X. Then L is positive if and only if it is ample.*

We will go through the proof in the simpler case in which X is a complex torus.

Proof. If X is a complex torus we have seen that positive implies ample in Lefschetz theorem. The converse is also true because if L is ample, then X is a subvariety of $\mathbb{P}^n$ via the map Ψ_{L^r} for some $r > 0$. Hence $(\Psi_{L^r})^*\omega_{FS} = \omega$ is the first chern class of L^r and it concludes the proof since $c_1(L^r) = r \cdot c_1(L)$. Therefore they represent the same cohomology class and positivity is preserved by the cohomology. □

In conclusion, we have two ways to establish whenever a complex torus is projective: we can look at its integer Kähler forms or we can look at ample line bundles. As expected, the links between these two approaches are the Riemann-Roch theorem, the Lefschetz theorem and the Kodaira embedding theorem.
In fact, roughly speaking, if a line bundle L^r induces an embedding we say by definition that L is ample, then it is positive by Kodaira theorem and so, by definition, its first class is definite positive and hence an integer Kähler form (we recall that by construction the form $c_1(L)$ is integer for any L). On the contrary if L induces an integer Kähler form, it is positive and by Riemann-Roch it has at least a global non null section. It may happen that the space of section (that has dimension $pf(\omega)$) is too small to allow the torus to be embedded in, but, since $c_1(L^r) = rc_1(L)$, it implies that $pf(c_1(L^r)) = pf(rc_1(L)) = r^g pf(c_1(L))$ and, by Riemann-Roch again, the bundle L^r has now "enough" sections to allow Ψ_L to be the embedding.
We can rephrase what we have said in terms of the notation of the Riemann Conditions theorem: if L is ample and $\omega_L = c_1(L)$, there exist positive integers $d_1|\dots|d_g$ such that the matrix associated with ω_L is as in Proposition 9.1. Since ω_L is definite positive, $d_1 \geq 1$ and it is the minimum possible value by construction. We notice that taking L^r implies that all d_i becomes d_i^r and hence the number of sections grow by Riemann-Roch. Actually one can prove without passing by sections that, if $d_1 \geq 2$, the map Ψ_L is holomorphic and, if $d_1 \geq 3$, the map Ψ_L is an embedding (Proposition 4.1.5 and 4.5.1 (BL)).
Finally let us notice that up to now we stated that to be projective implies to admit an integer Kähler form. Now we can say that the converse is also true because an integer Kähler form, since the map c_1 is surjective, is associated with a non degenerate positive line bundle and thus an ample line bundle thanks to Kodaira Theorem.

10.3 Polarized Abelian Varieties

Definition. An abelian variety is a complex torus that is a subvariety of $\mathbb{P}^n$ for some n.

It means that a complex torus is an abelian variety if and only if it admits an ample line bundle or, equivalently, it admits an integer Kähler form. Let us recall that this concepts are very related because if L is ample, hence $c_1(L)$ is an integer Kähler form.

Definition. A polarization on an abelian variety is an integer Kähler form. A polarized abelian variety is the couple (X, ω) where X is an abelian variety and ω is a polarization.

When we work on polarized abelian variety we will denote the morphism $\varphi_L \colon X \to \hat{X}$, its kernel $K(L)$ and the form e^L by, rispectively, φ_ω, $K(\omega)$ and e^ω since they depends only by the polarization.

An important case of polarization occurs when $\mathrm{pf}(\omega) = 1$, i.e. exactly when all $d_i = 1$. In this case we call (X, ω) principally polarized abelian variety and thanks to Proposition 9.8 the map φ_ω is an isomorphism since $\ker \varphi_L$ has cadinality equal to one, that is it contains just the zero. Moreover if L is a line bundle such that $c_1(L) = \omega$, Riemann-Roch theorem ensure that the vector space of holomorphic section has dimension 1 and then $|L|$ (the set of effective divisors associated with non null holomorphic sections of L) is composed by a unique element that we denote by Θ. Principally polarized abelian variety are very important thanks to this following result.

Proposition 10.7. *Any abelian variety is isogenic to a principally polarized abelian variety. More precisely, for any ample line bundle L on X there exist an abelian variety Y, a line bundle M defining a principal polarization on Y and a isogeny $u\colon X \to Y$ such that $L \cong u^*M$.*

Proof. Let $X = V/\Gamma$ an abelian variety and ω a polarization. Let us consider $(\gamma_1, \ldots, \gamma_{2g})$ the basis for Γ in which the matrix of ω is as in Proposition 9.1. Let Γ' the lattice generated by $(\gamma_1/d_1, \ldots, \gamma_g/d_g, \gamma_{g+1}, \ldots, \gamma_{2g})$. The form ω is hence integer and unimodular on Γ' and thus it defines a principal polarization on the torus $Y = V/\Gamma'$. The canonical surjection $u\colon X \to Y$ is an isogeny. Let L be a line bundle over X with $c_1(L) = \omega$ and M be a line bundle over Y with $c_1(M) = \omega$. The line bundles L and u^*M have the same first Chern class and hence they belong to the same $Pic^H(X)$. By Proposition 9.4 $L \cong \tau_x^* u^* M \cong u^* \tau_{u(x)}^* M$ and that concludes the proof. □

Even if the general complex torus is not a projective variety, the study of abelian variety is important because we can associate to every complex torus X an abelian variety X_{ab}.

Proposition 10.8. *Let X be a complex torus. There exists a complex torus X_{ab} (called the abelianization of X) and a holomorphic surjection $\rho\colon X \to X_{ab}$ such that:*

(1) There exists an integer Kähler form on X_{ab}, i.e. X_{ab} is an abelian variety.

(2) Every holomorphic function from X to a projective space factorises by ρ.

(3) ρ induces an isomorphism between the fields of meromorphic functions $\mathcal{M}(X_{ab})$ and $\mathcal{M}(X)$.

(4) ρ induces an isomorphism between the groups $Div(X_{ab})$ and $Div(X)$.

Proof. Let N be the intersection of the kernels of all Riemann forms on X. There exists a finite number of theta functions $\theta_1, \ldots, \theta_r$ such that $N = \ker \omega_1 \cap \cdots \cap \ker \omega_r$. Since ω_i are positive forms, $N = \ker(\omega_1 + \cdots + \omega_r)$ and $\omega_1 + \cdots + \omega_r$ is the form associated with the theta function $\theta_1 \cdots \theta_r$. We define $V_{ab} := V/N$ and, as we proved in Proposition 7.7, the image Γ_{ab} of Γ in V_{ab} is still a lattice and we can define the torus $X_{ab} = V_{ab}/\Gamma_{ab}$.
(1) The form $\omega := \omega_1 + \cdots + \omega_r$ is still integer on X_{ab} and is clearly non degenerate, so it is an integer Kähler form.
(2) Thanks to Proposition 7.11, any holomorphic map $u\colon X \to \mathbb{P}^n$ is made by normalized theta function of the same type, hence having the same Riemann form ω. By construction, $N \subseteq \ker \omega$ and u factorises by the quotient $\rho\colon X \to X_{ab}$ in the sense that there exists a holomorphic function $\tilde{u}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\rho"'] \arrow[rr, bend left, "u"] & X_{ab} \arrow[r, "\tilde{u}"'] & \mathbb{P}^n \end{tikzcd}$$

(3) We know that any meromorphic function f is quotient of two normalized theta function of the same type. Again by Proposition 7.7 we have that any such theta function comes from X_{ab}.
By the same reason (4) holds. In fact any divisor is difference of two effective divisors and any effective divisor is the divisor associated with a theta function. □

We take advantage of this proposition to make a digression about the meromorphic function field on an abelian variety and on a general complex torus.

Proposition 10.9. *Any line bundle on an abelian variety has a non null meromorphic section*

Proof. Let X be an abelian variety, L be a line bundle and M be an ample line bundle over X. Thanks to Riemann-Roch theorem, M has a non null section s. For $m \in \mathbb{Z}$ big enough, $c_1(L \otimes M^m)$ is positive definite and hence it has a non null section t. The section t/s^m is a non null meromorphic section of L □

Proposition 10.10. *The meromorphic function field of an abelian variety of dimension g is a finite extension of $\mathbb{C}$ with transcendence degree equal to g.*

Proof. See (Deb) Theorem 7.2. □

11 Digression: The "Very General" Complex Torus

In the previous section we gave a motivation to study abelian varieties but the great majority of complex tori is not projective. In this chapter we study their weird properties and we will gradually see how the "general" complex torus is different from an abelian variety. We call them "general" complex tori because we will show that complex tori with these following properties are a dense subset in the set of all complex tori. To formalise this, we classify tori by classifying the lattices: we parametrize them by the dense open set $U = GL(2g, \mathbb{R}) \subseteq \mathbb{R}^{4g^2}$. We denote by Γ_M the lattice that has for basis the columns of the matrix $M \in U$. Let us notice that up to now we are considering the data of a lattice with a basis.

We will find that all good properties of abelian varieties are encoded in a "small" subsets of $GL(2g, \mathbb{R})$, more precisely in the intersection of this space with a countable union of hypersurfaces of $\mathbb{R}^{4g^2}$. Anyway this result does not surprise us: it is quite natural to imagine that the existence of an integer Kähler form arises from a particular interaction between the lattice and the vector space.

During this section we will use the following lemma.

Lemma 11.1. *Let V be a vector space and $(Z_n)_n$ a countable set of proper algebraic hypersurfaces. Then the set of all vectors that do not belong to $\bigcup_n Z_n$ is dense in V.*

Proof. For all n, the set $V \setminus Z_n$ is open and dense. Moreover $V \setminus (\bigcup_n Z_n) = \bigcap_n (V \setminus Z_n)$ and this second set is certainly dense even if it may not be open. □

Theorem 11.2. *There exist a countable family $(Z_n)_n$ of real algebraic hypersurfaces of $\mathbb{R}^{4g^2}$ such that for all matrices $M \in U \setminus \bigcup_n Z_n$ the only complex subtori of $X = V/\Gamma_M$ are $\{0\}$ and X. We say that a general complex torus is simple.*

Proof. Let us notice that Y is a subtorus of X of dimension $k < g$ if and only if there exists a complex subspace $W \subseteq V$ of complex dimension k such that $\Gamma' := W \cap \Gamma$ is a lattice for W and $Y = W/\Gamma'$. In other words complex subtori of dimension k are in correspondence with complex vector subspaces generated by k vectors in Γ. In particular it happens when, given $\eta_1, \ldots, \eta_k \in \Gamma$,

$$rank\Big(\eta_1 \Big| \ldots \Big| \eta_k \Big| J\eta_1 \Big| \ldots \Big| J\eta_k\Big) = k$$

where J is the complex structure of V, that is the span of the η_j is closed under the action of the complex structure of V.
Let M be the matrix whose columns form a basis for a lattice Γ:

$$M = \Big(\gamma_1 \Big| \ldots \Big| \gamma_{2g}\Big) \in GL(2g, \mathbb{R}).$$

Let us consider k vectors $\eta_1, \ldots, \eta_k$ that are integer linear combination of the columns of M (and then elements of Γ). This set of vectors generates a subtorus of X if and only if the rank as above is equal to k. To check it we consider the sum of the square of all $k \times k$ minors and we want it to be zero, i.e. we want all minors to vanish. We denote by $Z_{n,k}$ the algebraic hypersurface of $\mathbb{R}^{4g^2}$ given by the zero locus of this polynomial equation. It is clear that varying the vectors, i.e. taking others $\eta'_1, \ldots, \eta'_k \in \Gamma$ we get an another algebraic hypersurface. Since we do not want X to have subtori of any dimension we must consider $M \in U \setminus \bigcup_{n,k} Z_{n,k}$ and by the construction we gave the set of all $Z_{n,k}$ is clearly countable.

□

This shows that to admit subtori is a very special and rare feature. We can try to figure out why it happens noting that, for example, the complex vector space $\mathbb{C}^2$ allows as vector subspaces only complex lines, while the real subspaces of dimension 2 of $\mathbb{R}^4$ are many more. Indeed if $(z = x+iy, w = \xi + i\eta)$ are the coordinates of $\mathbb{C}^2$, the complex lines are of the form

$$\lambda z + \mu w = 0 \iff \begin{cases} \lambda_1 x - \lambda_2 y + \mu_1 \xi - \mu_2 \eta = 0 \\ \lambda_1 y + \lambda_2 x + \mu_1 \eta + \mu_2 \xi = 0 \end{cases}$$

that clearly do not parametrize all 2-dimensional real vector spaces of $\mathbb{R}^4$.

We now pass to the most important question: the great majority of complex tori is not projective. To show this we will show that on a dense set of complex tori, integer Kähler forms are not allowed.

Theorem 11.3. *For all matrices $M \in U \setminus \bigcup_n Z_n$ all $\mathbb{R}$-bilinear alternating forms integer on Γ_M of type (1,1) are null.*

Proof. **STEP 1:** We want to study, given a vector space V and a linear subspace $Z = \{v \in V \mid F(v) = 0\}$ for some linear form F, how many lattices of V do not intersect Z.
Let us consider the lattice Γ_M parametrized by the matrix $M \in GL(2g, \mathbb{R})$ and denote by γ_j the j-th column of M. We define the linear subspaces

$$Z_{(a_1,\dots,a_{2g})} = \left\{ M \in \mathbb{R}^{4g^2} \mid F_{(a_1,\dots,a_{2g})}(M) := \sum_{k=1}^{2g} a_k F(\gamma_k) = 0 \right\}$$

where $(a_1, \dots, a_{2g}) \in \mathbb{Z}^{2g}$. The set of all $Z_{(a_1,\dots,a_{2g})}$ is clearly countable and the union of them represent exactly those lattices that intersects Z. We can then apply the lemma to say that a dense set of lattice does not intersect Z.
STEP 2: Let us now consider the vector space V of the real 2-forms on $\mathbb{R}^{2g}$, represented by antisymmetric matrices. Let Z be the linear subspace of V composed by $(1,1)$ forms. We have to build a lattice in V that parametrises integer forms on the lattice Γ_M in order to fit into the Step 1. Since $M \in GL(2g, \mathbb{R})$, we can consider the matrix ${}^tM^{-1}$ that represent the dual lattice Γ_M^*. We recall that an element in $\gamma^* \in \Gamma_M^*$ is a linear form integer on Γ_M, indeed $\gamma^* = \sum_{k=1}^{2g} m_k \gamma_k^*$, where $m_k \in \mathbb{Z}$ and $\gamma_k^*(\gamma_j) = \delta_{kj}$. We can then consider the lattice $\Lambda^2\Gamma_M^*$ generated by the vectors $\gamma_i^* \wedge \gamma_j^*$. We are now in the situation of the first step and it concludes the proof. □

Remark. If one is interested in an explicit description of V and Z appearing in the Step 2 of the proof we can notice that V is generated by the vectors $\{e^i \wedge e^j \text{ for } 1 \le i < j \le 2g\}$ and Z is the linear subset given by $Z = \{\alpha \in V \mid \alpha - J\alpha = 0\}$ where J is the complex structure induced on V by the complex structure of $\mathbb{R}^{2g}$ as in Lemma 3.2. We thus have to write the vectors $\gamma_i^* \wedge \gamma_j^*$ with respect to this basis:

$$\gamma_k^* \wedge \gamma_l^* = \sum_{1 \le i < j \le 2g} ({}^tM^{-1})_{klij}\, e^i \wedge e^j.$$

where $({}^tM^{-1})_{klij}$ is the determinant of the $2{\times}2$ matrix removing all rows except the i-th and the j-th and all columns except the k-th and the l-th. Therefore we have that, for $a_{ij} \in \mathbb{Z}$, the equation defining

Z becomes

$$
\begin{aligned}
0 &= \Big(\sum_{k,l} a_{kl} \gamma_k^* \wedge \gamma_l^* \Big) - J \Big(\sum_{k,l} a_{kl} \gamma_k^* \wedge \gamma_l^* \Big) = \\
&= \sum_{k,l} a_{kl} \Big(\gamma_k^* \wedge \gamma_l^* - J(\gamma_k^* \wedge \gamma_l^*) \Big) = \\
&= \sum_{k,l} a_{kl} \left(\sum_{1 \le i < j \le 2g} ({}^t M^{-1})_{klij}\, e^i \wedge e^j - J \sum_{1 \le i < j \le 2g} ({}^t M^{-1})_{klij}\, e^i \wedge e^j \right) = \\
&= \sum_{k,l} a_{kl} \sum_{1 \le i < j \le 2g} ({}^t M^{-1})_{klij} \Big[e^i \wedge e^j - J(e^i \wedge e^j) \Big].
\end{aligned}
$$

That has solution exactly when the 2×2 minors of M^{-1} are linearly dependent over $\mathbb{Z}$.

This last theorem shows us that for such tori $NS(X) = \{0\}$ and hence $Pic(X) = Pic^0(X)$. An another interesting consequence of this fact is that $c_1(L) = 0$ for every line bundle and hence necessarily $\dim \ker \omega = g$, $K(L) = X$ and $\varphi_L = 0$.

Corollary 11.4. *The general complex torus of dimension $g > 1$ is not projective.*

Proof. It is an easy consequence of the previous result and Theorem 5.2. Indeed a projective torus has an integer Kähler form given by the pullback of the Fubini-Study form and the pullback preserves its properties. □

General complex tori are hence an example of compact Kähler manifolds that are not projective.

Proposition 11.5. *For a general complex torus of dimension $g > 1$ all theta function are trivial and all meromorphic functions are constant.*

Proof. Thanks to Proposition 7.2 we know that we can associate to any theta function a $(1,1)$- form in $\Lambda^2\Gamma^* \cap \Lambda^{1,1}V^*$ and we know that this set is zero for the general complex torus, hence we have that $\omega = 0$ for all theta functions. We conclude thanks to Proposition 7.3. Since we know that any meromorphic function is quotient of two theta functions, they are actually holomorphic (since trivial theta functions never vanish) and thus constant. □

Notice that this shows that the field of meromorphic function of a general complex torus is $\mathbb{C}$ itself.

In the end of this section we try to put together all these results in order to point out the strange geometry of general complex tori studying divisors and sections of line bundles.

Will be useful to recall the notion of abelianisation: following the notation of Proposition 10.8, if X is a complex torus we have tree possibilities:

(1) $X = X_{ab}$

(2) $X \neq X_{ab}$ and $X_{ab} \neq \{0\}$

(3) $X_{ab} = \{0\}$

Where the case (1) means X is an abelian variety. Let us study more precisely the case (3). We discovered that if $X_{ab} = \{0\}$ we have that:

- All theta functions are trivial.
- All meromorphic functions are constant, i.e. $\mathcal{M}(X) = \mathbb{C}$.
- The following three equivalent properties hold

$$NS(X) = 0, \qquad Pic(X) = Pic^0(X), \qquad c_1 \equiv 0$$

11.1 Divisors

We can put all these information in the exact sequences we studied in the previous chapter and we get:

$$\begin{tikzcd}
 & & 0 \arrow[r] & H^1(\mathbb{Z}) \arrow[r] & \mathbb{C} \arrow[dr, bend left, two heads] & & & \\
0 \arrow[r] & \mathbb{C}^* \arrow[r, "\cong"] & \mathbb{C}^* \arrow[r] & 0 \arrow[r] & Div(X) \arrow[r, hook, "\Phi"] \arrow[drr, dotted, bend right, "\eta"'] & Pic^0(X) \arrow[r] \arrow[dr, bend right, "c_1 \equiv 0"] & H^1(\mathcal{M}^*) \arrow[r] & \dots \\
 & & & & & & H^2(\mathbb{Z}) \arrow[r] & \dots
\end{tikzcd}$$

Hence $\ker \Phi = \{0\}$ that means there are not principal divisors. We can say more:

Proposition 11.6. *If X is a complex torus such that $X_{ab} = \{0\}$,*

$$Div(X) = H^0(X, \mathcal{M}^*/\mathcal{O}^*) = \{0\}$$

Proof. It is an obvious consequence of Proposition 10.8 (4). □

Remark. If the reader is interested in an alternative and more direct proof one can recall that any divisor is difference of two effective divisors and any effective divisors are divisors associated with a theta function. Therefore, since all theta functions on X are trivial, the thesis follows.

11.2 Sections of Line Bundles

On general complex tori both holomorphic and meromorphic functions are constant. We would like to pull out some information in studying sections of line bundles. However we already know, thanks to Kodaira embedding theorem, that on such tori positive line bundles cannot exist, and hence Riemann-Roch theorem can not be used to study holomorphic sections. We will start to study what happens to section on those general tori whose abelianization X_{ab} is not trivial. To do so we have to generalise our results for positive semidefinite line bundles. Indeed, if $X_{ab} \neq \{0\}$, we will see in the next lemma that always there exists a positive semidefinite line bundle on X, because X_{ab} is projective and hence it allows a positive line bundle. Its pullback via the projection $p\colon X \to X_{ab}$ is the line bundle desired.

Lemma 11.7. *Let L be a line bundle on X and denote by $p\colon X \to X_{ab}$ the natural quotient projection. There exist a line bundle L_{ab} on X_{ab} with $L \cong p^* L_{ab}$ if and only if $L_{|K(L)}$ is trivial. If L_{ab} exists, it is nondegenerate and $dim H^0(X, L) = dim H^0(X_{ab}, L_{ab})$.*

Proof. By definition $X_{ab} = V_{ab}/\Gamma_{ab}$. The line bundle $L(H,\alpha)$ descends to X_{ab} if H descends to V_{ab} and α to Γ_{ab}, but this is the case if and only if $H_{|\Gamma(L)} = 0$ and $\alpha_{|\Gamma(L)\cap\Gamma} = 1$, i.e. $L_{|K(L)}$ is trivial.
If $L_{|K(L)}$ is trivial, L_{ab} is nondegenerate by construction and certainly $dimH^0(X,L) \geq dimH^0(X_{ab}, L_{ab})$. The equality holds because otherwise L would admit a section whose restriction to $K(L)$ is non trivial. □

Definition. If ω is an non null alternating semidefinite form, we denote by $\mathrm{Pfr}(\omega)$ its reduced pfaffian it is of the form

$$\mathrm{Pfr}(\omega) = \prod_{\nu=1}^{s} d_\nu$$

where $d_1|d_2|\dots|d_s$ and for $\nu > s$, $d_\nu = 0$ are the positive integers as in Proposition 9.1.

Theorem 11.8. *For a positive semidefinite line bundle $L = L(H,\alpha)$ on X it holds that:*

$$dimH^0(X,L) = \begin{cases} Pfr(\omega) & \text{if } L_{|K(L)} \text{ is trivial} \\ 0 & \text{if } L_{|K(L)} \text{ is nontrivial} \end{cases}$$

Proof. Let us suppose that L is trivial on $K(L)$. By the previous lemma, it descends to a non degenerate line bundle L_{ab} on X_{ab} with $dimH^0(X,L) = dimH^0(X_{ab}, L_{ab})$. Let us denote by ω_{ab} the form associated with it. By construction $Pf(\omega_{ab}) = Pfr(\omega)$ and hence by Riemann-Roch, $dimH^0(X,L) = Pfr(\omega)$ as desired.
Let us now suppose L is not trivial on $K(L)$, i.e. α restricts to a non trivial character on $\Gamma(L)\cap\Gamma$. Let θ be a canonical theta function of L. For any $w \in V$ the function $\tau_w^*\theta$ is holomorphic on V and satisfies

$$t_w^*\theta(v+\gamma) = \alpha(\gamma)t_w^*\theta(v)$$

for all $\gamma \in \Gamma(L)\cap\Gamma$ and $v \in \Gamma(L)$. Therefore $t_w^*\theta$ is bounded and hence constant. Moreover since α is not trivial, there exists a γ_0 such that $\alpha(\gamma_0) \neq 1$ and hence $t_w^*\theta = 0$ on $\Gamma(L)$. In particular $\theta(w) = 0$ and hence $\theta = 0$. □

We now wonder about what happens in the case X_{ab} is trivial. The previous approaches may be dangerous because in this case we have $X_{ab} = \{0\}$ and $K(L) = X$. Anyway we recall that Proposition 8.6 establishes a bijective correspondence between sections of $L = L(H,\alpha)$ and normalized theta functions of type (H,α). First of all let us notice that in this case $Pic(X) = Pic^0(X)$ and hence $H = 0$ for all L. Therefore all theta functions are trivial and it means that the space of normalized theta function of type $(0,\alpha)$ is composed by just one function. This implies that the vector space of sections of the line bundle has dimension 0 because it is composed only by the null section corresponding to the trivial theta function.
What about meromorphic sections? Proposition 8.3 says that a line bundle L admits a non trivial meromorphic section if and only if $L = \mathcal{O}_X(D)$ for some divisor D modulo principal divisor. In fact, if $D = (U_\alpha, h_\alpha)$ with h_α meromorphic, it turns out that (h_α) is a section of $\mathcal{O}_X(D)$. In our case $Div(X) = \{0\}$ and hence no line bundle admits a meromorphic section.

12 Moduli Space

In analogy with the one dimensional case the moduli space of all complex tori up to isomorphism is given by the quotient

$$GL(g,\mathbb{C})\backslash GL(2g,\mathbb{R})/GL(2g,\mathbb{Z}).$$

Unfortunately, unlike the one dimensional case, this space is not Hausdorff and hence it is not a topological manifold. We will thus reduce the set of complex tori whose we would like to study isomorphism classes. We will see through this section that the moduli space of polarized abelian varieties of the same type is an analytic manifold.

12.1 When are Abelian Varieties of the same Type Isomorphic?

Let $X = V/\Gamma$ be a complex torus and ω a polarization on X. We know that there exists a basis of Γ, namely $(\gamma_1, \dots, \gamma_{2g})$, in which the integer matrix associated with ω is

$$\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \quad \text{where} \quad \Delta = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix}.$$

$\Delta = (d_1, \dots, d_g)$ is the type of the polarization and we will say it is the type of the polarized abelian variety (X, ω).

Before going on let us make a recap about all the complex and real basis we are going to use:

- The $\mathbb{Z}$-basis $\mathcal{Z} = (\gamma_1, \dots, \gamma_{2g})$ of Γ in which ω has the form $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$.
- The real basis $\mathcal{B} = (\gamma_1, \dots, \gamma_g, e_1, \dots, e_g)$ of the vector space V, with $e_j = \gamma_{g+j}/d_j$ in which ω has the form $\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.
- The complex basis $\mathcal{C} = (e_1, \dots, e_g)$ of V and its real version $\mathcal{C} = (e_1, \dots, e_g, ie_1, \dots, ie_g)$.

Let us look at change basis matrix (occasionally we will see $\mathcal{Z}$ as a real basis of V).

- $\mathcal{M}^{\mathcal{Z},\mathcal{B}} = \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1}$
- $\mathcal{M}^{\mathcal{Z},\mathcal{C}} = \begin{pmatrix} Re\tau & \Delta \\ Im\tau & 0 \end{pmatrix}^{-1}$
- $\mathcal{M}^{\mathcal{B},\mathcal{C}} = \begin{pmatrix} Re\tau & I_g \\ Im\tau & 0 \end{pmatrix}^{-1}$

Remark. The integer numbers d_j are $\omega(\gamma_j, \gamma_{g+j})$. It is clear that two polarized abelian varieties (X, ω) and (X', ω') may have the same type Δ not being isomorphic.

As in the proof of the Riemann Conditions, we can consider the $\mathbb{C}$-basis of V given by $e_j = \gamma_{g+j}/d_j$ for $j = 1, \ldots, g$. The matrix of the components of the basis γ_j in this new basis is $(\tau\ \Delta)$ and hence any polarized abelian variety of type Δ is the quotient X_τ of $\mathbb{C}^g$ by the lattice $\Gamma_\tau = \tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g$, where τ belongs to the Siegel half space

$$\mathcal{H}_g = \{\tau \in \mathcal{M}_g(\mathbb{C}) \mid \tau = {}^t\tau,\ Im\tau > 0\}.$$

So any polarized abelian variety of type Δ determines a point in $\mathcal{H}_g$. Conversely, given Δ, any $\tau \in \mathcal{H}_g$ determines a polarized abelian variety of type Δ by $X_\tau = V/\tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g$. It is well defined because $\omega = ImH = (Im\tau)^{-1}$ is an integer Kähler form of matrix $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$. This means that there is a bijection between all possible polarized abelian varieties of type Δ and the space $\mathcal{H}_g$. Notice that here we are considering isomorphic abelian varieties as different.

The next step is to build a moduli space for isomorphism classes. A necessary and sufficient condition for the varieties $X_{\tau'}$ and X_τ to be isomorphic is that there exists an automorphism u of $\mathbb{C}^g$ such that $u(\Gamma_{\tau'}) = \Gamma_\tau$. We would like to study this condition in order to find a relation between τ' and τ in $\mathcal{H}_g$.

Remark. While surely does exist an automorphism v of $\mathbb{R}^{2g}$ such that $v(\Gamma_{\tau'}) = \Gamma_\tau$, it may happen that it is not a complex automorphism of $\mathbb{C}^g$, recall in fact that $GL(g, \mathbb{C}) \subsetneq GL(2g, \mathbb{R})$. Notice that it means that $v(\tau'\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g) = v(\tau'\mathbb{Z}^g) \oplus v(\Delta\mathbb{Z}^g) = \tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g$ because $\tau'\mathbb{Z}^g$ and $\Delta\mathbb{Z}^g$ generate real subspaces of V.

Let A be the matrix associated with the automorphism u in the canonical basis of $\mathbb{C}^g$ and let N its integer matrix in the basis of $\Gamma_{\tau'}$ and of Γ_τ given by the columns of the matrices

$$(\tau'\ \ \Delta) := \begin{pmatrix} Re\tau' & \Delta \\ Im\tau' & 0 \end{pmatrix} \quad \text{and} \quad (\tau\ \ \Delta) := \begin{pmatrix} Re\tau & \Delta \\ Im\tau & 0 \end{pmatrix}.$$

Let us define the automorphism of $GL(2g, \mathbb{Q})$

$$\sigma_\Delta \colon P \mapsto \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} P \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}.$$

Remark. The matrix that appears in σ_Δ is important because it holds that

$$\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^t\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} = \begin{pmatrix} I_g & 0 \\ 0 & \Delta^{-1} \end{pmatrix} \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^t\begin{pmatrix} I_g & 0 \\ 0 & \Delta^{-1} \end{pmatrix} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}. \tag{2}$$

It will be easier (in a sense that will be clear in the third section) to work with the matrix $M = \sigma_\Delta(N)$ in the basis C' and C corresponding to the columns of the matrices $(\tau'\ \ I_g)$ and $(\tau\ \ I_g)$. They are related by the identity

$$A(\tau'\ \ I_g) = (\tau\ \ I_g)M$$

and if we denote ${}^tM$ by the block matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the previous relations become

$$A\tau' = \tau\,{}^ta + {}^tb \quad \text{and} \quad A = \tau\,{}^tc + {}^td$$

that implies that the matrix $\tau^t c + {}^t d$ is invertible and

$$\tau' = {}^t\tau' = {}^t(\tau\,{}^t a + {}^t b)\,{}^t A^{-1} = (a\tau + b)(c\tau + d)^{-1}.$$

Here we do a second step: we do not want u to be just an isomorphism of abelian varieties, but we want that it preserves the polarizations. The complex isomorphism u induces an isomorphism of polarized abelian varieties (X, ω) and (X', ω') if $\omega = u^*\omega'$. In terms of matrices (in the basis C and C') this condition is equivalent to

$$M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} {}^t M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

hence M and ${}^t M$ belongs to the symplectic group $Sp(2g, \mathbb{Q})$.

Remark. If we recall that $M = \sigma_\Delta(N)$, the previous relation becomes

$$\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} N \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix} {}^t N \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} =$$
$$\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} N \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^t N \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} =$$
$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

where the last equality holds if and only if

$$N \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^t N = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}.$$

In conclusion: a necessary and sufficient condition for the polarized abelian varieties X_τ and $X_{\tau'}$ to be isomorphic is that there exists a matrix in

$$G_\Delta := \sigma_\Delta\Big(\mathcal{M}_{2g}(\mathbb{Z})\Big) \cap Sp(2g, \mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Q}) \mid a, b\Delta^{-1}, \Delta c, \Delta d \Delta^{-1} \text{ have integer coefficients} \right\}$$

such that:

- The matrix $c\tau + d$ is invertible (we will see that it is always true),
- $\tau' = (a\tau + b)(c\tau + d)^{-1}$.

The isomorphism between X_τ and $X_{\tau'}$ is of the form

$$\begin{array}{ccc} \mathbb{C}^g/\tau\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g & \longrightarrow & \mathbb{C}^g/\tau'\mathbb{Z}^g \oplus \Delta\mathbb{Z}^g \\ z & \longmapsto & {}^t(c\tau + d)^{-1} z \\ \tau p + q & \longmapsto & \tau' p' + q' \end{array}$$

where $\begin{pmatrix} p' \\ q' \end{pmatrix} = {}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$

12.2 Moduli Space of Polarized Abelian Varieties of the same Type

To build moduli spaces is not difficult, what is difficult is to build moduli space that shares some of the same properties with the geometric objects it parameterise. Our aim is to build this moduli space and to describe its analytic properties. In order to give it the structure of analytic space we will need Cartan theorem.

Theorem 12.1 (Cartan)**.** *Let X be an analytic space, G a group and $G \times X \longrightarrow X$ a properly discontinuous action via biholomorphisms. The ring sheaf $\mathscr{O}$ is defined for all open sets $U \subseteq X/G$ by*

$$\mathscr{O}(U) = \{f : U \longrightarrow \mathbb{C} \mid f \circ \rho \text{ is holomorphic in } \rho^{-1}(U)\},$$

where $\rho\colon X \longrightarrow X/G$ is the quotient projection, and it defines an analytic structure on X/G.

Let us remark that an analytic space is not a complex manifold because it may have some singular points. Anyway, under the hypothesis of Cartan theorem, singularities that may arise are not so bad and the space X/G inherits a lot of good properties of complex manifolds. If the action $G \times X \longrightarrow X$ is free, the quotient X/G is a complex manifold.

Let us now get to the heart of the question.

Proposition 12.2. *Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the map*

$$\begin{aligned} &G \times X \rightarrow X \\ &(M, \tau) \rightarrow M \cdot \tau = (a\tau + b)(c\tau + d)^{-1} \end{aligned}$$

defines a left action of $Sp(2g, \mathbb{R})$ on $\mathcal{H}_g$.

Proof. We will show that the action is well defined, that is $M \cdot \tau \in \mathcal{H}_g$. A matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the symplectic group if and only if

- ${}^t ac$ is symmetric,
- ${}^t bd$ is symmetric,
- ${}^t ad - {}^t cb = I_g = {}^t da - {}^t bc$.

Hence

$$\overline{{}^t(c\tau + d)}(a\tau + b) - \overline{{}^t(a\tau + b)}(c\tau + d) = \tau - \overline{\tau} = 2iIm\tau.$$

If there exists v such that $(c\tau + d)v = 0$, this formula implies that $\overline{{}^t v}(2iIm\tau)v = 0$ and thus $v = 0$ since $Im\tau$ is non degenerate. This implies that the action is well defined ($c\tau + d$ is inversible) for all $M \in Sp(2g, \mathbb{R})$. Let us denote $\tau' = (a\tau + b)(c\tau + d)^{-1}$. It holds that

$$\overline{{}^t(c\tau + d)}(\tau' - {}^t\tau')(c\tau + d) = \tau - {}^t\tau = 0$$

that means that τ' is symmetric. Moreover

$$\overline{{}^t(c\tau + d)}(\tau' - {}^t\overline{\tau'})(c\tau + d) = 2iIm\tau \tag{3}$$

that ensure us $Im\tau'$ is definite positive.
Hence $\tau' = M \cdot \tau \in \mathcal{H}_g$. It is easy to verify that $M \cdot \tau$ satisfies the axioms to be an action and this concludes the proof. □

Proposition 12.3. *The action of any discrete subgroup of $Sp(2g, \mathbb{R})$ on $\mathcal{H}_g$ is properly discontinuous.*

Proof. Let G be a discrete subgroup of $Sp(2g, \mathbb{R})$ and K be a compact set in $\mathcal{H}_g$. Let us suppose by contradiction that there is an element $M \in G$ such that $(M \cdot K) \cap K \neq \emptyset$, let τ_M be in this intersection and denote $\tau'_M := M^{-1} \cdot \tau_M$. Let H_{τ_M} be the definite positive Hermitian form with matrix $(Im\tau_M)^{-1}$ and let u_M be the automorphism of $\mathbb{C}^g$ with matrix ${}^t(c\tau_M + d)^{-1}$. The equation 3 can hence be rewritten as

$$H_{\tau'_M} = H_{\tau_M} \circ u_M.$$

Since $Im\tau_M$ and $Im\tau'_M$ belongs to a compact subset og $GL(g, \mathbb{C})$, so is for u_M. It follows that $c\tau_M + d$ is in a compact subset and so is for its imaginary part $cIm\tau_M$ and hence it holds for c and d. By the fact that $a\tau'_M + b = \tau_M(c\tau'_M + d)$, also a and b belongs to a compact set. Hence M is in a compact set of G and discrete compact sets in G are finite. □

Hence the isomorphism classes of polarized abelian varieties of type Δ is in bijection with the analytic space $G_\Delta \backslash \mathcal{H}_g$ of dimension $g(g+1)/2$: the same of $\mathcal{H}_g$ since a properly discontinuous action of a discrete group on a manifold gives a orbit space of the same dimension of the manifold.
We denote this moduli space by

$$\mathcal{A}_{g,\Delta} := G_\Delta \backslash \mathcal{H}_g.$$

Remark (Moduli Space of Elliptic Curves)**.** Let us recall the fact that any elliptic curve is a projective manifold and hence a polarized abelian variety of complex dimension $g = 1$. We saw that the moduli space of all elliptic curves up to isomorphism is

$$GL(1, \mathbb{C}) \backslash GL(2, \mathbb{R}) / GL(2, \mathbb{Z}).$$

We wonder if these isomorphism classes are also symplectomorhpism classes. To answer this question we need to have a concrete comprehension of what this double quotient is. Clearly $GL(2, \mathbb{R})$ parameterises all possible basis for all possible lattices. The action of $GL(2, \mathbb{Z})$ allows us to consider a lattice up to the choice of the basis and the action of $GL(1, \mathbb{C})$ allows us to put in the same class different lattices that give isomorphic tori. In the case $g = 1$ it happens that $GL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) = Sp(2, \mathbb{Z})$ that means that basis changes preserve the polarization, namely ω. Notice that the matrix of ω is of the form $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$ where $\Delta \in \mathbb{C}$ is the number $\Delta = \omega(\tau_1, \tau_2)$. Now, acting via $GL(1, \mathbb{C})$ we can consider the lattice $(\tau_1, \tau_2/\Delta)$ and it turns out that the indiced polarization is by construction principal. This means that any elliptic curve is isomorphic (but clearly not necessarily symplectomorphic) to a principally polarized elliptic curve.

When $g > 1$ it happens that $Sp(2g, \mathbb{Z}) \subsetneq GL(2g, \mathbb{Z})$, hence not any basis change preserves the polarization.

12.3 A Slightly Different Approach

When in the proof of Riemann Conditions we decided to introduce the basis $(e_1, \ldots, e_g)$ with $e_j = \gamma_{g+j}/d_j$, we knew we were making a choice in the approach of the construction of moduli space of polarized abelian varieties. In fact in such a basis we have

$$\omega(\gamma_j, e_j) = \omega(\gamma_j, \gamma_{g+j}/d_j) = \frac{1}{d_j}\omega(\gamma_j, \gamma_{g+j}) = \frac{1}{d_j}d_j = 1,$$

hence the matrix of ω is $J := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$.

This is why, in the previous section, we studied the standard symplectic group, i.e. the linear automorphisms that preserve the matrix J. This is not strictly necessary: in fact we could continue to work with the basis $(\gamma_1, \dots, \gamma_{2g})$ and study the group

$$Sp^{\Delta}(2g, \mathbb{R}) = \left\{ P \in GL(2g, \mathbb{R}) \;\middle|\; P \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^tP = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \right\}.$$

In fact the map σ_Δ we defined establishes an isomorphism between $Sp^{\Delta}(2g, \mathbb{R})$ and $Sp(2g, \mathbb{R})$. It is clarely an automorphism of $GL(2g, \mathbb{R})$, we just prove that if $P \in Sp^{\Delta}(2g, \mathbb{R})$, hence $\sigma_\Delta(M) \in Sp(2g, \mathbb{R})$:

$$\begin{aligned}
&\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} P \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} {}^t\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix} {}^tP \, {}^t\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} = \\
&= \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} P \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^tP \, {}^t\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} = \\
&= \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} {}^t\begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} = \\
&= \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.
\end{aligned}$$

Remark. The group $Sp^{\Delta}(2g, \mathbb{R})$ is not invariant under transposition.

Given a matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp^{\Delta}(2g, \mathbb{R})$, we can thus define the action

$$P \cdot \tau := (a\tau + b\Delta)(\Delta^{-1}c\tau + \Delta^{-1}d\Delta)^{-1}$$

Remark. If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp^{\Delta}(2g, \mathbb{R})$, we have that

$$\sigma_\Delta(P) = \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} I_g & 0 \\ 0 & \Delta \end{pmatrix} = \begin{pmatrix} a & b\Delta \\ \Delta^{-1}c & \Delta^{-1}d\Delta \end{pmatrix} \in Sp(2g, \mathbb{Q})$$

Remark. By the fact that

$$G_\Delta := \sigma_\Delta\Big(\mathcal{M}_{2g}(\mathbb{Z})\Big) \cap Sp(2g, \mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Q}) \;\middle|\; a, b\Delta^{-1}, \Delta c, \Delta d \Delta^{-1} \text{ have integer coefficients} \right\}$$

we can conclude that

$$G_\Delta = \sigma_\Delta\Big(Sp^{\Delta}(2g, \mathbb{Z})\Big).$$

A consequence of these remarks is that the space $\hat{\mathcal{A}_{g,\Delta}} := Sp^{\Delta}(2g, \mathbb{Z}) \backslash \mathcal{H}_g$ is isomorphic to the space $\mathcal{A}_{g,\Delta}$ defined before.

12.4 Why Principally Polarized Abelian Varieties are Important?

Complex tori arise very often in complex geometry and their importance is due to the fact that all complex compact Lie groups are tori. We saw that abelian varieties are just a very special example of complex tori because the very general complex torus is not projective. Anyway also abelian varieties arise in a very incredible and important way being the only example of projective tori. Going further we studied polarized abelian varieties and, in particular, principally polarized abelian varieties that are, as we have seen, a special example of polarized abelian varieties. One may wonder why one should be interested in this extremely special case of complex tori. Here we collect some motivations. First of all we recall Proposition 10.7.

Proposition. *Any abelian variety is isogenic to a principally polarized abelian variety. More precisely, for any ample line bundle L on X there exist an abelian variety Y, a line bundle M defining a principal polarization on Y and a isogeny $u\colon X \to Y$ such that $L \cong u^*M$.*

This means that in some way principally polarized abelian varieties encode a lot of the geometry of a general abelian variety.

The second motivation we give is the historical motivation of the study of these objects. It arises from the study of complex projective curves, in particular Torelli's theorem 12.8 gives an interesting criterion to establish when two compact Riemann surfaces of genus g are isomorphic. Let C be a smooth projective curve of genus g over the field of complex numbers. Let us denote by $H^0(\omega_C)$ the g-dimensional complex vector space of holomorphic 1-forms on C. Moreover recall that the homology group $H_1(C,\mathbb{Z})$ is a free abelian group of rank $2g$. We notice that any $\gamma \in H_1(C,\mathbb{Z})$ can be seen as a linear form on $H^0(\omega_C)$ by integration:

$$\omega \mapsto \int_\gamma \omega.$$

Lemma 12.4. *There is a canonical injection*

$$H_1(C,\mathbb{Z}) \to H^0(\omega_C)^* := Hom\big(H^0(\omega_C),\mathbb{C}\big).$$

Proof. We recall some facts: by the universal coefficient theorem, there is a canonical immersion $H_1(C,\mathbb{Z}) \to H_1(C,\mathbb{C}) = H^1_{DR}(C)^*$. Moreover we have the Hodge decomposition $H^1_{DR}(C) = H^{1,0}(C) \oplus H^{0,1}(C) = H^0(\omega_C) \oplus \overline{H^0(\omega_C)}$. Since the dual of the direct sum is the direct sum of the duals, we are looking for the composition map

$$H_1(C,\mathbb{Z}) \to H^1_{DR}(C)^* \xrightarrow{p} H^0(\omega_C)^*$$

where p is the projection. Since the image of any $\gamma \in H_1(C,\mathbb{Z})$ is invariant under complex conjugation, it is necessarily of the form $l + \bar{l}$ with $l \in H^0(\omega_C)$ and this implies the assertion. □

It follows that $H_1(C,\mathbb{Z})$ is a lattice in $H^0(\omega_C)^*$.

Definition. The Jacobian variety of a smooth projective curve of genus g is the complex torus

$$J(C) := \frac{H^0(\omega_C)^*}{H_1(C,\mathbb{Z})}.$$

What happens if $C = X = \mathbb{C}/\Gamma$ is an elliptic curve, i.e. a complex algebraic curve of genus $g = 1$?

- The space $H^0(\omega_X)^*$ is $\mathbb{C}$,
- The space $H_1(C,\mathbb{Z})$ is Γ.

Hence for an elliptic curve $J(X) \cong X$. In particular $J(X)$ is a complex torus of dimension 1. There is an important theorem that generalises this fact.

Theorem 12.5. *(Abel-Jacobi) Let C be a smooth projective curve of genus g over the field of complex numbers. The Jacobian variety is a torus of complex dimension g and it holds that*

$$J(C) \cong Pic^0(C).$$

Idea: The first assertion is clear by the definition of $J(C)$. The isomorphism $J(C) \cong Pic^0(C)$ is explicitly given by the Abel-Jacobi map. Recall that for an algebraic curve $Pic^0(C)$ is the quotient of degree 0 divisors, namely $Div^0(C)$ by principal divisors and that any $D \in Div^0(C)$ is of the form $D = \sum_{j=1}^{N}(p_j - q_j)$ with points $p_j, q_j \in C$. The Abel-Jacobi map is

$$\begin{aligned} Div^0(C) &\longrightarrow H^0(C)^* \\ D &\mapsto \left\{\omega \mapsto \sum_{j=1}^{N} \int_{q_j}^{p_j} \omega\right\} \mod H_1(C,\mathbb{Z}) \end{aligned}$$

We do not prove it is an isomorphism mod $H_1(C,\mathbb{Z})$. For all details see (BL), Theorem 11.1.3. □

Remark. It is coherent with the fact that for an elliptic curve (that is a projective curve of genus $g = 1$) its Jacobian has dimension 1.

The most incredible fact is that the lattice $H_1(C,\mathbb{Z})$ and the vector space $H^0(\omega_X)^*$ interact in such a perfect way that any Jacobian variety is a principally polarised abelian variety. We want to build directly the principal polarization: in a first step we study how the complex structure $\tilde{I}$ of C induces a complex structure I on $H^0(\omega_C)^*$, then we define a 1-form on $H^0(\omega_C)^*$ invariant under $H_1(C,\mathbb{Z})$. Finally we prove that this form, in a suitable basis of $H_1(C,\mathbb{Z})$ is a principal polarization for $J(C)$, in particular we prove that it is an integer Kähler form that is positive definite and with $\Delta = 1_g$.
The first step is given for free: the Hodge decomposition we used in the proof is strictly related to the complex structure of C and if we change structure, the Hodge decomposition also changes. Moreover it is clear, since $H^1_{DR}(C)$ is a vector space, that giving its decomposition is the same as giving a complex structure. By theory of complex vector spaces, it is induced a complex structure on the dual and on any vector subspace and hence on $H^0(\omega_C)^*$.
To define the form we need, we use intersection theory. We will just set up the theoretical tools in our contest. Roughly speaking we have a complex projective curve of genus g and we want to "count" intersections of its 1-submanfolds. We want to define something well defined up to homology classes, hence we will "count" intersections in a suitable way. On C we have a fixed complex structure (the one we used to define the Hodge decomposition) and we know it fixes a orientation for C, i.e. a basis for any tangent plane such that the basis change between charts has positive determinant.

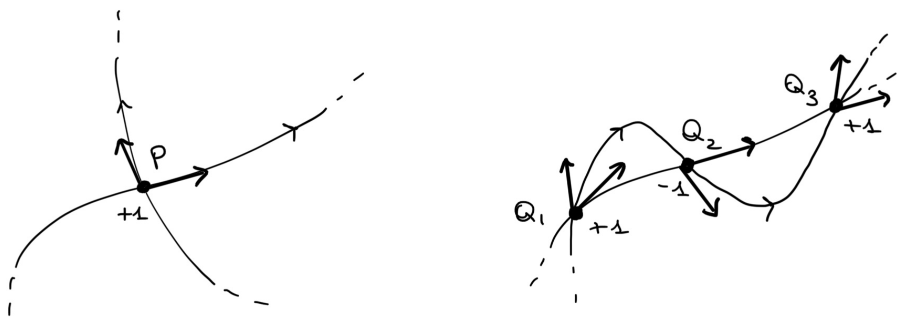
Definition. Given two curves $\gamma(t)$ and $\eta(t)$ in C, we say that the point $P \in C$ is a positive (negative) intersection if exist t_0 and t_1 such that $\gamma(t_0) = \eta(t_1)$ and the vectors $\dot{\gamma}(t_0), \dot{\eta}(t_1)$ induce the same (opposite) orientation as the complex structure on the tangent plane.

Remark. It may happen that γ and η intersect in a point P with the same tangent. This makes impossible to establish whenever P is negative or positive. Luckily we want to work up to homology classes and there is a result that ensure that if γ and η intersect with the same tangent there exist $\gamma' \in [\gamma]$ and $\eta' \in [\eta]$ such that the intersection is transverse. In a certain sense it is clear because we know that "little" perturbations of curves do not change the homology class.

Definition. Given two curves $\gamma(t)$ and $\eta(t)$ in C, we define the number of intersections as

$$\Omega(\gamma,\eta) = \sum_{P \text{ pos.}} 1 - \sum_{P \text{ neg.}} 1 \in \mathbb{Z}$$

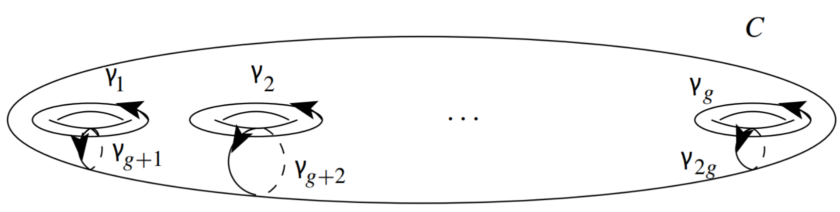
We will not prove that it is well defined on homology classes, that is, if $\gamma' \in [\gamma]$ and $\eta' \in [\eta]$, $\Omega(\gamma',\eta') = \Omega(\gamma,\eta)$, but we will see through a drawing that it makes sense.



Here it is clear that if the point Q_1 is positive, so is Q_3 and Q_2 is negative. In particular in both cases the number of intersections counted as we defined is equal to 1.
We can now define the intersection form

$$\begin{aligned} \tilde{\Omega}_C \colon H_1(C,\mathbb{Z}) \times H_1(C,\mathbb{Z}) &\to \mathbb{Z} \\ ([\gamma],[\eta]) &\longmapsto \Omega(\gamma,\eta). \end{aligned}$$

If we extend this form by linearity we get a form Ω_C on $H^0(\omega_C)^*$ that is integer on $H_1(C,\mathbb{Z})$, hence it is an integer Kähler form on the torus $J(C)$ if and only if it is of type $(1,1)$, i.e. $\Omega_C(I[\gamma], I[\eta]) = \Omega_C([\gamma],[\eta])$. We want more: we want Ω_C to be a principal polarization. Let us consider the basis $\gamma_1,\dots,\gamma_{2g}$ of $H_1(C,\mathbb{Z})$ as shown in the following figure.



(BL), pg. 317.

It is easy to check that $\gamma_1,\dots,\gamma_{2g}$ form a basis and that in this basis the matrix of Ω_C is

$$\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

In conclusion we have to prove that

(1) Ω_C is of type $(1,1)$,

(2) Ω_C is positive.

We recall that these conditions are equivalent to the following properties of the bilinear form H associated with Ω_C:

(1) H is Hermitian,

(2) H is positive definite.

In fact conditions (2) are equivalent by definition, since

$$H(u,v) = \overline{H(v,u)} \iff \Omega_C(u,Iv) - i\Omega_C(u,Iv) = \Omega_C(v,Iu) + i\Omega_C(v,Iu)$$

where the last equality clearly holds if and only if Ω_C is of type (1,1).

Lemma 12.6. *The form Ω_C is of type $(1,1)$.*

Proof. We have to understand how the complex structure I acts on classes $[\gamma]$ and how it affects the intersection product. Indeed we have that I acts on the vector space $H^0(\omega_C)^*$, whose vectors are real linear combination of $[\gamma_i]$, we wonder how curves in $[\gamma]$ differs form curves in $I[\gamma]$.
To do so let us consider the class $[\gamma] \in H_1(X,\mathbb{R})$. Thanks to Poincaré duality we can consider the class $[\gamma^*] \in H^1(X,\mathbb{R})$ and work on this space. We stress the fact that Poincaré duality also establish a relation between the wedge product and the intersection product. This is very helpful because there is a canonical way to give differential forms a complex structure induced by the one on the algebraic curve. Let us consider on the curve the coordinate $z = x + iy$. We recall that given a differential form, the complex structure acts as $I\alpha(\cdot,\dots,\cdot) := \alpha(I\cdot,\dots,I\cdot)$ where, with an abuse of notation, we are calling with I two different complex structures. Therefore it is induced the complex structure that locally acts this way $Idx = -dy$ and $Idy = dx$. One can show that this complex structure is actually compatible with the wedge product, and it is crucial for us because we want it to be compatible with the intersection product when we pull it down to homology. Let $\alpha = \alpha_1 dx + i\alpha_2 dy$ and $\beta = \beta_1 dx + i\beta_2 dy$:

$$\alpha \wedge \beta = i(\alpha_1\beta_2 - \alpha_2\beta_1)\, dx \wedge dy$$

$$I\alpha \wedge I\beta = (-\alpha_1 dy + i\alpha_2 dx) \wedge (-\beta_1 dy + i\beta_2 dx) = i(\alpha_1\beta_2 - \alpha_2\beta_1)\, dx \wedge dy$$

showing the compatibility.
Finally to get that the intersection form is of type $(1,1)$ it suffices to recall that

$$\Omega_C(I[\gamma], I[\eta]) = \int_C I\gamma^* \wedge I\eta^* = \int_C \gamma^* \wedge \eta^* = \Omega_C([\gamma],[\eta]).$$

□

We need, finally, the following lemma. We will not prove it.

Lemma 12.7. *The Hermitian form H is positive definite.*

It follows that the form Ω_C is a principal polarization and hence the complex manifold $(J(C),\Omega_C)$ is a principally polarized abelian variety, as desired.

The importance of studying Jacobian varieties comes from the following result.

Theorem 12.8 (Torelli). *Suppose C and C' are compact Riemann surfaces of genus g. If their Jacobians $(J(C), \Omega)$ and $(J(C'), \Omega')$ are isomorphic as polarized abelian varieties, then C is isomorphic to C'.*

A mathematical question that have been active for years concerning Jacobian varieties is the Schottky problem. The statement of the problem is the following: which principally polarised abelian varieties of dimension g arise from Jacobian varieties of complex projective curves of genus g? More precisely, we have the moduli space, namely $\mathcal{M}_g$, of algebraic curves of genus g and the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties. The problem consists in studying the map

$$\mathrm{Jac}_g \colon \mathcal{M}_g \to \mathcal{A}_g.$$

Torelli's theorem implies it is injective.
Riemann made a new formulation of the problem based on period matrices. In this thesis we have seen the Riemann conditions to establish which lattice generates a projective torus. One can build the period matrix of a complex projective curve of genus g, formed by integrating a basis for the abelian integrals round a basis for the first homology group and the question becomes which period matrices satisfies the Riemann conditions. Takahiro Shiota solved the problem in this new formulation in 1986. We will not go through the details, but it is interesting to face the problem just in a dimensional way: we know that

$$dim\mathcal{M}_g = 3g - 3 \quad \text{and} \quad dim\mathcal{A}_g = g(g+1)/2.$$

Therefore dimension coincides when $g = 0, 2, 3$.. Dimension 0 is a limit case and it is not in our interest.
Dimension 1 is easy since genus 1 curve are elliptic curves and we have seen that if X is such a curve $J(X) \cong X$, hence, trivially, Jac_1 is an isomorphism.
For $g > 3$, $g(g+1)/2 > 3g - 3$ and hence surely there are principally polarized abelian varieties that are not the Jacobian of any curve.
For $g = 2, 3$ also the easy dimensional problem is interesting. One can show that for $g = 2$ we have two type of principally polarized abelian varieties: the Jacobians of genus 2 curves and product of Jacobians of genus 1 curves, that is product of elliptic curves. This construction hold also in the case $g = 3$ since a principally polarized abelian variety of dimension $g = 3$ may be not the jacobian of any curve but being product of smaller Jacobians.

13 Moduli Space of Hyperkähler Structures on a Complex Torus

13.1 Hyperkähler Manifolds

In October 1843 math's history passed through the Broom Bridge, when Sir William Rowan Hamilton discovered the formula for quaternions multiplication:

$$i^2 = j^2 = k^2 = ijk = -1.$$

We will denote the non commutative division algebra of quaternions by $\mathbb{H}$. An element of this algebra is of the form $x + iy + jz + kw$ with $x, y, z, w \in \mathbb{R}$. As in $\mathbb{C}$, the operation of conjugation is defined:

$$\overline{x + iy + jz + kw} = x - iy - jz - kw.$$

More than one hundred years later, Eugenio Calabi in 1978 gived the definition of an Hyperkähler manifold, very related to quaternions.

Definition. Let (M, g) be a Riemannian manifold of real dimension $4n$ equipped with three complex structure operators $I, J, K \colon TM \to TM$ satisfying the quaternionic relation. Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is an Hyperkäler manifold.

Remark. The definition implies that g is the Kähler metric for all the complex structures. This means that

$$g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y).$$

We know that Kähler and symplectic geometry are very related because the Kähler form is a real symplectic form. This relation extends in a natural way on Hyperkähler manifolds: trivially they are real symplectic manifolds by taking the three Kähler forms

$$\omega_I(X, Y) := g(IX, Y) \qquad \omega_J(X, Y) := g(JX, Y) \qquad \omega_K(X, Y) := g(KX, Y)$$

but they are also holomorphically symplectic manifolds when considering the forms

$$\Omega_I := \omega_J + i\omega_K \qquad \Omega_J := \omega_K + i\omega_I \qquad \Omega_K := \omega_I + i\omega_J$$

that are clearly closed and holomorphic differential forms with values in $\mathbb{C}$. One can do more putting all information in a unique differential form with values in the quaternion algebra:

$$\Omega := i\omega_I + j\omega_J + k\omega_k \in \Omega^2(M, \mathbb{H}).$$

Remark. The 2-form Ω can not be defined as "symplectic" because it take values on $\mathbb{H}$ that is not a field, but it is clearly closed and non degenerate.

All these constructions implies that locally our manifold "seems" $\mathbb{H}^n$. More precisely, its tangent spaces are free left $\mathbb{H}$-modules. As in real and complex geometry we have scalar and Hermitian products, also in quaternionic geometry we can build something similar and study which matrix Lie group preserves it.

Definition. The quaternionic Hermitian product is

$$((p,q)) = pq^\dagger = \sum p_i \overline{q}_i,$$

where $q^\dagger = {}^t\overline{q}$. This decomposes as above in the $n = 1$ case:

$$((\cdot,\cdot)) = dq \otimes dq^\dagger = g - i\omega_I - j\omega_J - k\omega_K = (g - i\omega_I) - j(\omega_J + i\omega_K) = H_I - j\Omega_I,$$

where H_I is the Hermitian form related to the form ω_I.

Definition. The lie group

$$Sp(n) := \{A \in M_n(\mathbb{H}) \mid AA^\dagger = \mathbb{I}\}$$

is the quaternionic unitary group.

Remark. Matrices in $Sp(n)$ are those which preserve the quaternionic Hermitian product. In the case of our interest the following isomorphisms hold:

$$Sp(1) \cong SU(2) \cong U(1,\mathbb{H})$$

Notice that preserving the quaternionic Hermitian product implies to preserve both the Hermitian form and the symplectic form. It follows that the quaternionic unitary group of $\mathbb{H}^n \cong \mathbb{C}^{2n} \cong T^*\mathbb{C}^n$ is

$$Sp(n) = SU(2n) \cap Sp(2n,\mathbb{C}).$$

13.2 Extremal Volume

The extremal volume is a generalisation of the concept of extremal length that arises in conformal geometry. Let us define extremal length and then we will do some comments. Let Λ be a family of curves on the open domain $\mathcal{D} \subseteq \mathbb{C}$ and let $\rho\colon \mathcal{D} \to \mathbb{R}$ a positive integrable function. It implies that $\rho^2 g_{std}$ is a conformal metric on $\mathcal{D}$. We set

$$l(\Lambda,\rho) := \inf_{\gamma\in\Lambda} \int_\gamma \rho|dz|, \qquad A(\rho) := \int_{\mathcal{D}} \rho^2 dxdy.$$

Notice that we have no hypothesis on Λ. We will see that some interesting cases may arise when it is a homology class.

Definition. The extremal length of Λ is

$$\mu_\Lambda := \sup_\rho \frac{l(\Lambda,\rho)^2}{A(\rho)}.$$

The result is a number between $[0,\infty]$ and to avoid the fact that μ_Λ is zero because the denominator is infinite, we restrict to those ρ which induce a positive finite area.
Roughly speaking we are studying which is the biggest length of the smallest curve in Λ when varying the conformal metric and with respect to the area of the domine. We clearly want that this quantity to be scale invariant, and it is trivially achieved because taking $c\rho$ instead of ρ returns a factor c^2 both to numerator and denominator.
We point out in addition that, how the notation suggest, μ_Λ depends only by Λ and not by $\mathcal{D}$, in the

sense that for all other $\mathcal{D}'$ such that $\Lambda \subseteq \mathcal{D}'$, μ_Λ remains the same. To see this, suppose that $\mathcal{D} \subseteq \mathcal{D}'$. Given ρ on $\mathcal{D}$ we can choose ρ' on $\mathcal{D}'$ such that $\rho' = \rho$ on $\mathcal{D}$ and $\rho' = 0$ on $\mathcal{D}' \setminus \mathcal{D}$. This proves that $\mu_\Lambda^{\mathcal{D}'} \geq \mu_\Lambda^{\mathcal{D}}$. For the opposite inequality we need only start from a ρ' on $\mathcal{D}'$ and let ρ be its restriction to $\mathcal{D}$.

We want to extend this concept to higher dimensions: the idea is to consider a complex manifold (M, J) of *complex* dimension n and the volume of its submanifolds of *real* dimension n. Notice that in order to work in complex dimension 1 we just need a Riemannian metric because it induces a volume form and because in complex dimension 1 conformal geometry and complex analysis are strictly related (for example, a holomorphic map $f\colon U \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if it is conformal). In higher dimension this relationship falls. The idea behind to extremal volume is to try to delete all Riemannian informations in order to build a theory that leans only on complex informations. In this way, we do not need to involve conformal geometry using just the complex structure.
As happens for the extremal length, we want to calculate volumes with respect to the "same" form. Therefore we have to find a suitable way to restrict a holomorphic n-form of M to a real submanifold of real dimension n. Let us denote by K_M the holomorphic line bundle of differential forms of type $(n, 0)$ and let Ω be a smooth section of it. We can build a real $2n$-form

$$\Omega_M := (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}.$$

We focus our attention on the fact that we do not require K_M to be trivial and hence we allow Ω to vanish in some points, anyway, if Ω is nowhere vanishing, Ω_M is a real volume form and K_M is differentiably trivial. Let us consider $p \in M$ and $T_pM \cong \mathbb{C}^n$ a complex tangent space. Let $\pi \subseteq T_pM$ be a real subspace of real dimension n. We can define a multilinear form on π in the following way:

$$\Omega_\pi(v_1, \ldots, v_n) := |\Omega(p)(v_1, \ldots, v_n)|,$$

where $(v_1, \ldots, v_n)$ is a real positive basis for π. It vanishes in two cases:

- $\Omega(p) = 0$,
- $v_1, \ldots, v_n$ are not complex linearly independent, i.e. π contains a complex line.

In conclusion, given a real submanifold L of M of real dimension n, we get a real n-form Ω_L on L setting

$$\Omega_L(p) := \Omega_\pi, \quad \text{where} \quad \pi = T_pL \quad \text{for any} \quad p \in L.$$

Our setting is complete:

- we call $A(\Omega) := \int_M \Omega_M$ the Ω-volume of M
- we call $\int_L \Omega_L$ the Ω-volume of L.

We recall the fact that Ω is allowed to vanish somewhere, as long as it is not identically zero.

Definition. Given a set Λ of real oriented submanifolds $L \subseteq M$ of dimension n, we let

$$l(\Lambda, \Omega) := \inf_{L \in \Lambda} \int_L \Omega_L$$

denote the infimum value of the Ω-volume functional restricted to Λ.

It is important to notice that if we take a function $f: M \to S^1$, the differential form $\Omega' := f\Omega$ returns the same value. We will say that Ω and Ω' are equivalent if they differ for a rotation.

Definition. The extremal volume of Λ is

$$\mu_\Lambda := \sup_\Omega \frac{l(\Lambda, \Omega)^2}{A(\Omega)}$$

where we restrict our attention to those admissible Ω such that $0 < A(\Omega) < \infty$.

As for the extremal length, we have that for any constant c, it holds that

$$\frac{l(\Lambda, \Omega)^2}{A(\Omega)} = \frac{l(\Lambda, c\Omega)^2}{A(c\Omega)},$$

and hence it is invariant under rescalings.
By the definition we gave, our invariant is non zero on totally real submanifolds that are those submanifold L of (M, J) such that for all $p \in L$, $J_p(T_pL) \cap T_pL = \{0\}$. Roughly speaking, totally real submanifolds are those submanifolds whose tangent spaces do not contain any complex line.

Remark. In our case, since we are considering submanifolds of real dimension n, the previous condition is equivalent to $J_p(T_pL) \oplus T_pL = T_pM$.

Remark. Let us notice that the construction we made of the extremal volume strongly depends on the complex structure we fixed at the beginning. This is exactly what we want, because the aim is to build some invariant that allow us to distinguish different complex structures.

Remark. We are trying to encode informations of a complex manifold using just complex differential forms and real submanifolds. In particular in our context of complex tori it assume a very deep importance because we have seen that the very general complex torus is simple and it even does not admit divisors. On the other hand we know that any complex torus is of the form V/Γ and hence it admits real submanifolds and real subtori.

The extremal volume is far to be simple to calculate because it involves a sup of an inf. We would like to find some conditions that would allow us to make simpler the calculation. Even if this theory holds for any complex manifold, our final goal is to apply it to tori, therefore we will introduce some further conditions that will be achieved by tori.

Definition. Let (M, J) be a complex manifold of complex dimension n and fix a smooth section Ω of K_M. An oriented real n-plane π is Ω-special if $\Omega_\pi = \Omega_{|\pi}$. We say that a submanifold L of M is Ω-special if each T_pL is Ω-special, i.e. $\Omega_L = \Omega_{|L}$

Looking at the definition of the form Ω_π, it is clear that a real n-plane π is Ω-special if, on that plane, Ω takes real values, or, equivalently, its imaginary part vanishes. Let us notice that this condition is trivially achieved when $\pi \subseteq T_pM$ is any subspace of real dimension n and $\Omega(p) = 0$. Therefore we will look only at the case $\Omega(p) \neq 0$ and π totally real.

Concerning oriented submanifolds and assuming Ω never vanishing (and then Ω_M is a volume form) we have to deal with different situations:

- L is a complex submanifold, then it is special for any Ω and it holds that $\Omega_L = 0$. Notice that this implies that $\int_L \Omega_L = 0$ and then L minimizes the Ω_L-volume compared to any other oriented submanifold.

- L is such that any tangent space T_pL contains a complex line. The consequences are the same of the previous case, in particular $\int_L \Omega_L = 0$.
- L is totally real. We can define a phase $e^{i\theta}\colon L \longrightarrow S^1$ such that $\Omega_L = e^{i\theta}\Omega$. Therefore L is special if and only if $e^{i\theta} = 1$. In this case we say that L is special totally real.

We can now prove an important result about a lower bound of the extremal volume. It is in the particular case in which $\Lambda = \alpha \in H_n(M, \mathbb{Z})$ is a homology class. It will be the main interesting case in the next section.

Proposition 13.1. *Let Ω be closed. Then any Ω-special submanifold L minimizes the Ω-volume functional within its homology class α. In particular $l(\alpha, \Omega) = \int_L \Omega$. This implies the lower bound*

$$\mu_\alpha \geq \frac{\left(\int_L \Omega\right)^2}{\int_M \Omega_M}.$$

Proof. Let $[L] = [L'] = \alpha$. Using the fact that all integrals are real we get

$$\int_L \Omega_L = \int_L \Omega = \int_{L'} \Omega \leq \int_{L'} |\Omega| = \int_{L'} \Omega_{L'}$$

and the equality holds exactly when L' is Ω-special.
To compute μ_α we must consider all admissible forms Ω', but since μ_α is scale invariant we can always achieve $A(\Omega) = A(\Omega')$ and the lower bound is thus immediate. □

13.3 Extremal Volume and Complex Tori

Up to now we have just a lower bound, while our goal is to find a way to calculate the extremal volume in case of complex tori. We notice that, in the case of the extremal length, an easy computation occurs when working with quadrilaterals, because the Riemann Mapping theorem allows us to restrict to the special case of rectangles, which have the property of being fibred by segments parallel to their sides. This use of quadrilaterals suggests to generalize this construction to complex tori. Let X be a complex torus of complex dimension g. Any homology class $\alpha \in H_g(X, \mathbb{Z})$ can be represented by a subtorus L generated by g $\mathbb{R}$-linearly independent vectors in the lattice and we wonder if we can generalize the fibration mentioned above.
We recall the following theorem.

Theorem 13.2. *Assume that, for some closed and admissible Ω, M admits a parallel special totally real fibration, with generic fibre L. Let α be the holomogy class of the fibre. Then*

$$\mu_\alpha = \frac{\left(\int_L \Omega\right)^2}{\int_M \Omega_M}.$$

Proof. For all details we refer to (TP2), Theorem 4.4. □

This result provides a very concrete way to compute extremal volume in this particular situation. This section is devote to show that this results applies to complex tori. First of all recall the following

Definition. A fibred manifold is a triple (X, B, f), where X and B are smooth manifolds and $f: X \to B$ is a surjective submersion. B is called the basis of the fibration and we denote by L the typical fibre, in the sense that $f^{-1}(b) \cong L$ for all $b \in B$.

Example 13.1. We focus our attention on the case that we will use in the next section. Notice that this construction, *mutatis mutandis*, holds in every dimension. Let $X = \mathbb{C}^2/\Gamma$ a complex torus with $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \rangle_{\mathbb{Z}}$. The fibration given by parallel translation of the typical fiber $L \cong \mathbb{R}^2/\Gamma''$, where $\Gamma'' = \langle \gamma_3, \gamma_4 \rangle_{\mathbb{Z}}$, is induced by $f: X \to B$, with:

- $B = \mathbb{R}^2/\Gamma'$ and $\Gamma' = \langle \gamma_1, \gamma_2 \rangle_{\mathbb{Z}}$,
- f is induced by the function

$$\begin{array}{rccc} \tilde{f} & : & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\ & & (x, y, z, w) & \longmapsto & (x, y) \end{array}$$

Proposition 13.3. *Let X be a complex torus of dimension g and B a real torus of real dimension g as in Example 13.1 inducing a parallel totally real fibration. Then the fibration is special.*

Proof. Since tori are a compact Lie groups, up to normalization we can deduce that any holomorphic $(n, 0)$-form is a constant rotation of $dz^1 \wedge \cdots \wedge dz^g$. Since the extremal volume is invariant under rotations, we can choose $e^{i\theta} \in \mathbb{C}$ such that $\Omega = e^{i\theta} dz^1 \wedge \cdots \wedge dz^g$ and the fibration is Ω-special. This last assertion is due to the fact that the fibration is parallel and hence if $b, b' \in B$ are different points there will exists a unique θ such that Ω_{L_b} and $\Omega_{L_{b'}}$ are equal to $e^{i\theta} dz^1 \wedge \cdots \wedge dz^g$ restricted respectively to L_b and $L_{b'}$, concluding the proof. $\square$

Thanks to Proposition 13.3, we can conclude that, in the particular case of the torus, in order to compute the extremal length of a homology class α it suffices a parallel totally real fibration and the standard volume form $\Omega = dz_1 \wedge \cdots \wedge dz_g$. Indeed we can consider $e^{i\theta}$ such that $e^{i\theta}\Omega = |\Omega|$ and the given fibration is therefore trivially $|\Omega|$-special. Therefore it holds that

$$\mu_\alpha = \frac{\left(\int_L |\Omega|\right)^2}{\int_X \Omega_X}.$$

This result makes even simpler to compute the extremal volume because, unlike in Theorem 13.2, we have not to choose a particular volume form, but we can just use the standard volume form.

13.4 Moduli Space

The aim of this section is to build the moduli space of all possible Hyperähler structures on a marked complex torus using the extremal volume. The idea of combining extremal volume and Hyperkähler structures on a complex torus arises from essentially four facts:

- First of all we want the extremal volume to be computable. We have seen that we want a parallel special totally real fibration and complex tori allow it. Moreover, being this construction very rigid and being tori the only example of compact complex Lie groups, we think that complex tori are a very interesting example of complex manifolds admitting such structure.

- A lot of moduli spaces concerning complex tori parametrize complex structure of abelian varieties. This is because abelian varieties, in contrast with general tori, have a lot of complex informations, being $Div(X)$ and $Pic(X)$ not trivial. By the use of extremal volume we can point out information from the general complex torus studying how the complex structure interacts with its real submanifolds.

- Since extremal volume is an invariant that depends only on the complex structure, working with Hyperkähler manifolds allows us to get a set of invariants for each complex structure and we can hope that this number of invariants is sufficient to build a moduli space.

- Working on complex tori gives us two advantages. The first is that being Lie groups we will see how the calculation of the extremal volume is further simplified. Moreover we proved a theorem that help us to calculate extremal volume when working on homology classes and homology of tori is very simple and concrete.

Remark. We have worked only on 2-dimensional complex tori, a further development should be to improve techniques to study higher dimensional cases.

Definition. Let (X, I, J, K, g) be a 2-dimensional complex Hyperkähler torus and set

$$\Omega_I := \omega_J + i\omega_K \qquad \Omega_J := \omega_K + i\omega_I \qquad \Omega_K := \omega_I + i\omega_J$$

We recall that any 2-torus inherits the standard complex structure of $\mathbb{R}^4$. In particular if we call (x, y, ξ, η) the coordinates of $\mathbb{R}^4$ we get:

- The complex structure $I_{std} \simeq (x + iy, \xi + i\eta)$ and the form $\Omega_I = (dx + idy) \wedge (d\xi + id\eta)$.

- The complex structure $J_{std} \simeq (x + i\xi, \eta + iy)$ and the form $\Omega_J = (dx + id\xi) \wedge (d\eta + idy)$.

- The complex structure $K_{std} \simeq (x + i\eta, y + i\xi)$ and the form $\Omega_K = (dx + id\eta) \wedge (dy + id\xi)$.

Notice that the complex structures are chosen to induce the same orientation.
In particular, given $\Gamma \in GL(4, \mathbb{R})$ the Hyperkähler torus is $(\mathbb{R}^4, I_{std}, J_{std}, K_{std}, g_{std})/\Gamma$. We know that the moduli space of 2-dimensional Hyperkähler tori can be written as:

$$Sp(1)\backslash GL(4, \mathbb{R})/SL(4, \mathbb{Z}).$$

Let us count the parameter of the moduli space:

$$dim\mathcal{T}_{HK} = 16 - 3 = 13$$

and if we further identify rescaled tori we can go one dimension down, getting

$$\tilde{\mathcal{T}}_{HK} := \mathcal{T}_{HK}/\mathbb{R}^+$$

whose dimension is 12.

Remark. This number is quite good for our research because it is even and moreover a multiple of 4, then we can hope to give the moduli space some complex, Kähler or Hyperkähler structure.

We notice that we can give a explicit interpretation to this 12 parameters. In fact the action of $\mathbb{R}^+ \times Sp(1)$ allows us map the first column of Γ to the vector $(1,0,0,0)$. In other words we may assume

$$\Gamma \simeq \begin{pmatrix} 1 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 & c_3 \\ 0 & a_4 & b_4 & c_4 \end{pmatrix}.$$

We can now go into the heart of the question and study if extremal volumes computed for each complex structure and for each homology class gives us a way to fully rebuild the starting torus (or lattice). In particular we know that any homology class of a torus can be represented by a subtorus and since we are interested in real 2 dimensional subtori we have exactly 6 homology classes

$$\alpha_{1,2} = \Big[\langle(1,0,0,0),(a_1,a_2,a_3,a_4)\rangle\Big], \ldots, \alpha_{3,4} = \Big[\langle(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle\Big]$$

where the class $\alpha_{i,j}$ is the class generated by the columns i and j, namely Γ_i and Γ_j, of the matrix representing Γ. Therefore, having three complex structures, we get 18 extremal volumes, namely $\mu^I_{\alpha_{i,j}}, \mu^J_{\alpha i,j}, \mu^K_{\alpha_{i,j}}$ for $i,j = 1,\ldots,4$.
What we will prove is that the rank of the function

$$\begin{array}{rccc} \mu & : & \mathbb{R}^{12} & \longrightarrow & \mathbb{R}^{18} \\ & & (a_i,b_i,c_i)_i & \mapsto & (\mu^I_{\alpha_{i,j}}, \mu^J_{\alpha_{i,j}}, \mu^K_{\alpha_{i,j}})_{i,j}, \end{array}$$

for $i,j = 1,\ldots,4$, is maximal and hence it is an immersion of the moduli space in $\mathbb{R}^{18}$.

Remark. As we have already said, to calculate extremal volumes of tori is very simple. It is due to the fact they are Lie groups. Specifically:

$$\mu^I_{i,j} = \frac{\left(\int_{\alpha_{i,j}} |\Omega_I|\right)^2}{\int_X \Omega_I} = \frac{\left|\int_{\alpha_{i,j}} \Omega_I\right|^2}{\int_X \Omega_I} = \frac{\left|\int_0^1 \int_0^1 \varphi^*\Omega_I\right|^2}{\int_X \Omega_I} = \frac{\left|\Omega_I(\Gamma_i,\Gamma_j)\right|}{\det\Gamma},$$

where $\varphi(s,t) = s\Gamma_i + t\Gamma_j$.
We want to prove the second equality. We recall that $|\Omega_I| = e^{i\theta}\Omega_I$.

$$\mathbb{R}^+ \ni \int_L |\Omega_I| = \int_L e^{i\theta}\Omega_I = e^{i\theta}\int_L \Omega_I = |e^{i\theta}| \cdot \left|\int_L \Omega_I\right| = \left|\int_L \Omega_I\right|$$

where the third equality holds since we are dealing with a real positive quantity.

This remark implies that extremal volumes related to homology classes are polynomial fractions, hence easy to compute. Actually to test this first case in complex dimension 2, we resorted to MAPLE, a software of symbolic computation: we made him compute the 18 extremal volumes and the jac obian J of the map μ. Then we asked it to compute it rank and the answer was 12. We have to explain this result: in fact the answer $rank(J) = 12$ implies that there is at least one point $(\bar{a}_i, \bar{b}_i, \bar{c}_i)_i \in \mathbb{R}^{12}$ such that the rank of J evaluated in this point is 12. We want to prove that this implies that $p \in \mathbb{R}^{21}$ such that $rank(J_p) = 12$ is a dense open set of $\mathbb{R}^{12}$. We know it is non empty and its complementary, that is all determinant of 12×12 minors are identically zero, is an algebraic hypersurface of $\mathbb{R}^{12}$ and hence the thesis follows. In the next pages there is the MAPLE code for the rank calculation.

```
> restart; with(VectorCalculus) : with(LinearAlgebra) :
      with(Student[VectorCalculus]) : with(DifferentialGeometry) :
```

$$> O\_I := \begin{bmatrix} 0 & 0 & 1 & I \\ 0 & 0 & I & -1 \\ -1 & -I & 0 & 0 \\ -I & 1 & 0 & 0 \end{bmatrix}:$$

$$> O\_J := \begin{bmatrix} 0 & I & 0 & 1 \\ -I & 0 & 1 & 0 \\ 0 & -1 & 0 & I \\ -1 & 0 & -I & 0 \end{bmatrix}:$$

$$> O\_K := \begin{bmatrix} 0 & 1 & I & 0 \\ -1 & 0 & 0 & -I \\ -I & 0 & 0 & 1 \\ 0 & I & -1 & 0 \end{bmatrix}:$$

We denote by L the matrix of the lattice and by Li its columns.

$$> L := \begin{bmatrix} 1 & a1 & b1 & c1 \\ 0 & a2 & b2 & c2 \\ 0 & a3 & b3 & c3 \\ 0 & a4 & b4 & c4 \end{bmatrix}:$$

```
> simplify(Determinant(L));
```

$$(a2\,b3 - a3\,b2)\,c4 + a3\,b4\,c2 + a4\,(b2\,c3 - b3\,c2) - a2\,b4\,c3 \qquad \textbf{(1)}$$

```
>
>
> L1 := Vector[row]([1, 0, 0, 0]) :
> L2 := Vector[row]([a1, a2, a3, a4]) :
> L3 := Vector[row]([b1, b2, b3, b4]) :
> L4 := Vector[row]([c1, c2, c3, c4]) :
```

Extremal volumes with complex structure I:

> $mI12 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L2, O\_I)))^2}{Determinant(L)}$:

> $mI13 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L3, O\_I)))^2}{Determinant(L)}$:

>

> $mI14 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L4, O\_I)))^2}{Determinant(L)}$:

> $mI23 := \frac{evalc(\mathrm{abs}(BilinearForm(L2, L3, O\_I)))^2}{Determinant(L)}$:

>

> $mI24 := \frac{evalc(\mathrm{abs}(BilinearForm(L2, L4, O\_I)))^2}{Determinant(L)}$:

>

> $mI34 := \frac{evalc(\mathrm{abs}(BilinearForm(L3, L4, O\_I)))^2}{Determinant(L)}$:

>

Extremal volumes with complex structure J:

> $mJ12 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L2, O\_J)))^2}{Determinant(L)}$:

>

>

$$mJ13 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L3, O\_J)))^2}{Determinant(L)}:$$

>

>

$$mJ14 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L4, O\_J)))^2}{Determinant(L)}:$$

>

>

$$mJ23 := \frac{evalc(\mathrm{abs}(BilinearForm(L2, L3, O\_J)))^2}{Determinant(L)}:$$

>

>

$$mJ24 := \frac{evalc(\mathrm{abs}(BilinearForm(L2, L4, O\_J)))^2}{Determinant(L)}:$$

>

>

$$mJ34 := \frac{evalc(\mathrm{abs}(BilinearForm(L3, L4, O\_J)))^2}{Determinant(L)}:$$

>

Extremal volumes with complex structure K:

>

> $$mK12 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L2, O\_K)))^2}{Determinant(L)}:$$

>

>

$$mK13 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L3, O\_K)))^2}{Determinant(L)}:$$

>

>

$$mK14 := \frac{evalc(\mathrm{abs}(BilinearForm(L1, L4, O\_K)))^2}{Determinant(L)} :$$

>

>

>

$$mK23 := \frac{evalc(\mathrm{abs}(BilinearForm(L2, L3, O\_K)))^2}{Determinant(L)} :$$

>

>

$$mK24 := \frac{evalc(\mathrm{abs}(BilinearForm(L2, L4, O\_K)))^2}{Determinant(L)} :$$

>

>

$$mK34 := \frac{evalc(\mathrm{abs}(BilinearForm(L3, L4, O\_K)))^2}{Determinant(L)} :$$

>

>

>

Denote by J the jacobian of the map from R^12 to R^18 given by extremal volumes.

> $J := Jacobian([mI12, mI13, mI14, mI23, mI24, mI34, mJ12, mJ13, mJ14, mJ23, mJ24, mJ34, mK12, mK13, mK14, mK23, mK24, mK34], [a1, a2, a3, a4, b1, b2, b3, b4, c1, c2, c3, c4]) :$

> $Rank(J);$

$$12 \qquad (2)$$

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